Finite volume discretization

Gergely Kristóf

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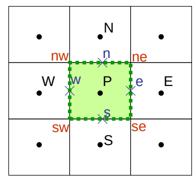
Numerical integration of the fluxes and the volume sources

$$\oint_A \rho\phi\underline{v}\cdot d\underline{A} = \oint_A \Gamma\nabla\phi\cdot d\underline{A} + \int_V q_\phi dV$$

convective flux conductive flux volume source

$$F_e = \int_A \underline{f}\cdot d\underline{A} = \langle f_\perp \rangle_e A_e \cong \frac{1}{2}\left(\underline{f}_P + \underline{f}_E\right)_\perp A_e$$ 2-nd order accurate

Compass notation:



Alternative surface integration schemes:

$$F_e \cong A_e \frac{1}{2}\left(\underline{f}_{ne} + \underline{f}_{se}\right)_\perp$$ 2-nd order accurate (trapeze method)

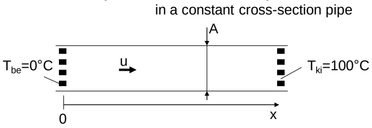
$$F_e \cong \frac{A_e}{6}\left(\underline{f}_{ne} + 4\underline{f}_e + \underline{f}_{se}\right)_\perp$$ 4-th order accurate (Simpson formula)

$$Q_P \cong \int_V q_\phi dV \cong q_{\phi,P} V_P$$ 2-nd order accurate

Interpolation of the fluxes must be at least as accurate as the integration scheme.

Application in 1D

Steady flow of a constant density fluid with heat conduction in a constant cross-section pipe

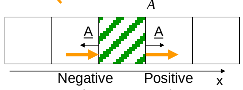


$$\underline{v} = \begin{pmatrix} u \\ v \\ w \end{pmatrix} \begin{matrix} \\ \leftarrow 0 \\ \leftarrow 0 \end{matrix}$$

Continuity: $\oint_A \rho\underline{v}\cdot d\underline{A} = 0 \longrightarrow \frac{\partial \rho u}{\partial x} = 0 \xrightarrow{\rho=\text{const.}} u = \text{constant}$

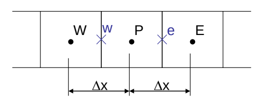
Energy equation: $\oint_A [c_v T + \cancel{\frac{u^2}{2}}]\rho\underline{v}\cdot d\underline{A} = \oint_A \lambda\nabla T\cdot d\underline{A}$

Applied for one cell: **by assuming fluxes in +x direction**



Discretization

$$\oint_A \rho u T\cdot dA_x = \oint_A \frac{\lambda}{c_v}\frac{\partial T}{\partial x}\cdot dA_x$$



The numerical integral of fluxes: $$(\rho u T)_e \cancel{A} - (\rho u T)_w \cancel{A} = \left(\frac{\lambda}{c_v}\frac{\partial T}{\partial x}\right)_e \cancel{A} - \left(\frac{\lambda}{c_v}\frac{\partial T}{\partial x}\right)_w \cancel{A}$$

Shorthand notations: $C_e = C_w = \rho u \qquad D_e = D_w = \frac{\lambda}{c_v\,\Delta x}$

$$C_e T_e - C_w T_w = D_e(T_E - T_P) - D_w(T_P - T_W)$$

... in a more simple form: $F_e - F_w = 0$,

in which: $F_e = C_e T_e - D_e(T_E - T_P)$ in the total flux.

In a 3D case we would have 4 more F values.

Application of the CDS scheme

$$C_e T_e - C_w T_w = D_e(T_E - T_P) - D_w(T_P - T_W)$$

Face temperatures (T_e *and* T_w) are obtained by a **linear interpolation**:

$$\left[\frac{C_e}{2}(T_P + T_E) - D_e(T_E - T_P)\right] - \left[\frac{C_w}{2}(T_W + T_P) - D_w(T_P - T_W)\right] = 0$$

The resultant linear equation for T_P:

$$A_P T_P = A_W T_W + A_E T_E$$

A_W	A_E	A_P
$D_w + C_w/2$	$D_e - C_e/2$	$A_W + A_E$

$$D_e + D_w + C_e/2 - C_w/2 = A_E + A_W + \underbrace{C_e - C_w}_{=0 \text{ kontinuitás}}$$

Since $A_P = A_W + A_E$, the linear equation for A_P can be regarded as a weighted average of the neighboring T values. T_P cannot be an extreme value, if the „A" values are positive.

Solution of the system of linear algebraic equations

For 4 cells:

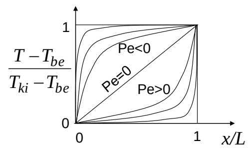
TA() TB() TC() TD()

$$\begin{bmatrix} -A_{1,P} & A_{1,E} & 0 & 0 \\ A_{2,W} & -A_{2,P} & A_{2,E} & 0 \\ 0 & A_{3,W} & -A_{3,P} & A_{3,P} \\ 0 & 0 & A_{4,W} & -A_{4,P} \end{bmatrix} \cdot \begin{bmatrix} T_{1,P} \\ T_{2,P} \\ T_{3,P} \\ T_{4,P} \end{bmatrix} = \begin{bmatrix} -A_{1,W}T_{be} \\ 0 \\ 0 \\ -A_{4,E}T_{ki} \end{bmatrix}$$

We can solve this system by Gauss elimination.
The matrix of the linear system is a tridiagonal matrix which requires only 2n operations in the case of n cells.
(This special case of the Gauss elimination is called the Thomas algorithm).

Unfortunately, such an efficient direct solution is not possible in 2D and 3D (iterative methods must be applied).

Analytical solution



$$\frac{T-T_{be}}{T_{ki}-T_{be}}=\frac{e^{\rho u x c_v/\lambda}-1}{e^{\rho u L c_v/\lambda}-1}$$

Pe (Peclet number)

Implementation in Excel macro

1. Similar solution is obtained with different input parameters.
2. The error reduces with N^2. (Second order accuracy.)
3. Sometimes the solution oscillates. What is the condition for the onset of instabilities?

$$Pe=\frac{\rho u L}{\lambda / c_v}$$

$$Re=\frac{\rho u L}{\mu}$$

$$Pe_{\Delta x}=\frac{\rho u \Delta x}{\lambda / c_v}>2$$

Transportivity

By physical means:
T_E must have a decreasing affect on T_P for an increasing value of Pe, because the heat conduction is overridden by the adverse convective flux.
Does the numerical scheme behaves so?

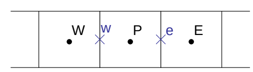
$$A_E=D_e-C_e/2$$

$$C_e=\rho u \qquad D_e=\frac{\lambda}{c_v\,\Delta x} \qquad Pe=\frac{\rho u L}{\lambda / c_v}$$

$$A_E=\frac{D_e}{2}\left(2-\frac{C_e}{D_e}\right)=\frac{D_e}{2}\left(2-\frac{\rho u \Delta x}{\lambda / c_v}\right)=\frac{D_e}{2}\left(2-Pe_{\Delta x}\right)$$

The cell Peclet number is the ratio of convective and conductive heat fluxes.
In the case of $Pe_{\Delta x}$>>2 the value of A_E can be a very large negative value.
This is not sensible from physical point of view.
This case is also numerically unstable.

Upwind Differencing Scheme (UDS)



for u>=0: $T_w=T_W,\quad T_e=T_P$

for u<0: $T_w=T_P,\quad T_e=T_E$

$$A_W T_W+A_E T_E=A_P T_P$$

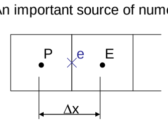
A_W	A_E	A_P
$Max(C_w,0)+D_w$	$Max(-C_e,0)+D_e$	A_W+A_E

Further numerical experiments...

Accuracy reduced to 1-st order.

Artificial diffusion

An important source of numerical errors. It came from the inaccurate interpolation:



$$T_e=T_P+\frac{\Delta x}{2}\frac{dT}{dx}+o(\Delta x)$$

we neglect these

$$F_e=C_e T_P+C_e\frac{\Delta x}{2}\frac{dT}{dx}-D_e(T_E-T_P)$$

It is like if the heat conductivity grew.
Let's substitute the numerical approximation of the temperature gradient:

$$\frac{dT}{dx}=\frac{T_E-T_P}{\Delta x}$$

$$F_e=C_e T_P+\frac{C_e}{2}(T_E-T_P)-D_e(T_E-T_P)$$

$$D_e=\frac{\lambda}{c_v\Delta x} \longrightarrow \frac{\lambda_{artif.}}{c_v\Delta x}=\frac{\rho u}{2} \longrightarrow \lambda_{artif.}=\frac{\rho u c_v \Delta x}{2}$$

Hybrid Differencing Scheme (HDS)

by Spalding (1972)
The positivity of the "A"s must be ensured.
We need to apply unwinding only if the absolute value of $Pe_{\Delta x}$ is too high.:

$Pe_{\Delta x}\le -2 \qquad F_e=C_e T_E$

$-2<Pe_{\Delta x}\le 2 \qquad F_e=C_e\left[\frac{1}{2}\left(1+\frac{2}{Pe_{\Delta x}}\right)T_P+\frac{1}{2}\left(1-\frac{2}{Pe_{\Delta x}}\right)T_E\right]$

$2<Pe_{\Delta x} \qquad F_e=C_e T_P$

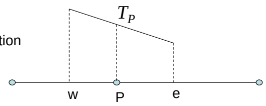
It is of second order accuracy for conduction dominated problems. (For small $Pe_{\Delta x}$ cases.)

$$A_W T_W+A_E T_E=A_P T_P$$

A_W	A_E	A_P
$Max\left(C_w,\left[D_w+\frac{C_w}{2}\right],0\right)$	$Max\left(-C_e,\left[D_e-\frac{C_e}{2}\right],0\right)$	A_W+A_E

Second Order Upwinding (SOU)

We can interpolate T within the simulation cell by using its gradient:



Wall fluxes than can be than evaluated like: $$T_e = T_P + \left.\frac{dT}{dx}\right|_P \frac{\Delta x}{2}$$

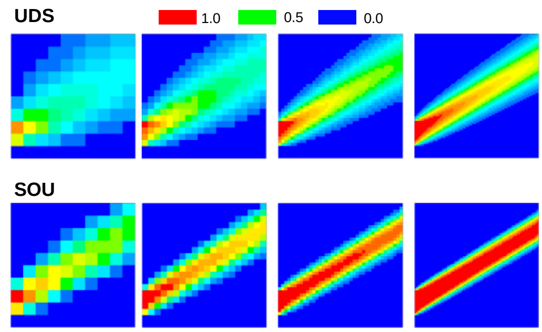
Gradients are calculated in 2 steps:

Firstly: $$\left.\frac{dT}{dx}\right|_P = \frac{T_e' - T_w'}{\Delta x} \qquad T_e' = \frac{T_P + T_E}{2}, \qquad T_w' = \frac{T_W + T_P}{2}$$

Secondly: $\left.\frac{dT}{dx}\right|_P$ gradients are limited on such a way that they shouldn't introduce oscillations. For details on the gradient limiters please refer: C Hirsch, Numerical computation of internal and external flows.

The numerical diffusion in practice

2D heat transport with zero heat conductivity (λ=0).



Mesh size: 10x10 20x20 40x40 80x80