# Advanced Fluid Mechanics

BME GEÁT MW01

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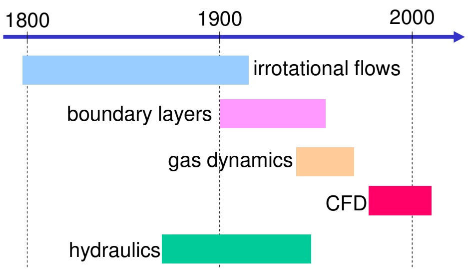
January, 2009.

## References

1) Lamb H: Hydrodynamics, 1932.
2) Schlichting H: Boundary Layer Theory, 1955.
3) Shapiro A. H: The Dynamics and Thermodynamics of Compressible Fluid Flow, 1953.
4) Streeter V. L, Wylie E. B: Fluid Mechanics, McGraw-Hill, 1975.
5) Ferziger J. H, Peric M: Computational Methods for Fluid Dynamics, Springer, ISBN 3-540-42074-6, 2002.

## Timeline




## 1. Introduction, review of vortical flows

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## Foreseeable program

| Course week | Topics |
|---|---|
| 1. | Vorticity transport equation. Irrotational flows. |
| 2. | Solution methods based on analytical solutions. |
| 3. | Darcy flow, airfoils. |
| 4. | Boundary layers. Similarity solutions for laminar and turbulent boundary layers. |
| 5. | Overview of computational fluid dynamics (CFD). |
| 6. | Turbulence models. |
| 7. | Fundaments of gas dynamics. Wave phenomena. |
| 8. | Isentropic flow, Prandtl-Meyer expansion, moving expansion waves. |
| 9. | Normal shock waves, oblique shock waves, wave reflection. |
| 10. | Jets. |
| 11. | Open surface flows, channel flows. |
| 12. | Pipe networks. |
| 13. | Transient flow in pipelines. |
| 14. | Atmospheric flows. |

## Acceleration of a fluid parcel

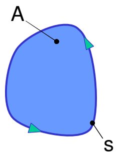
Velocity components: $\vec{v}(t,\vec{r}) = u(t,x,y,z)\vec{i} + v(t,x,y,z)\vec{j} + w(t,x,y,z)\vec{k}$

$$du = \frac{\partial u}{\partial t}dt + \frac{\partial u}{\partial x}dx + \frac{\partial u}{\partial y}dy + \frac{\partial u}{\partial z}dz$$

For a fluid parcel: $\frac{dx}{dt} = u, \quad \frac{dy}{dt} = v, \quad \frac{dw}{dt} = w$

$$\left.\begin{aligned} \frac{du}{dt} &= \frac{\partial u}{\partial t} + \vec{v}\cdot\nabla u \\ \frac{dv}{dt} &= \frac{\partial v}{\partial t} + \vec{v}\cdot\nabla v \\ \frac{dw}{dt} &= \frac{\partial w}{\partial t} + \vec{v}\cdot\nabla w \end{aligned}\right\} \rightarrow \frac{d\vec{v}}{dt} = \underbrace{\frac{\partial \vec{v}}{\partial t}}_{\text{local acceleration}} + \underbrace{\vec{v}\cdot\overbrace{\nabla\vec{v}}^{\text{velocity gradient tensor}}}_{\text{convective acceleration}}$$

# Vortices




Circulation:

$$\Gamma = \oint_s \vec{v} \cdot d\vec{s}$$

Vorticity:

$$\vec{\omega} = \nabla \times \vec{v} = \begin{pmatrix} w'_y - v'_z \\ u'_z - w'_x \\ v'_x - u'_y \end{pmatrix}$$

From the Stokes theorem we also know that:

$$\Gamma = \int_A \vec{\omega} \cdot d\vec{A}$$

Thomson: If s is a fluid line of a perfect fluid, then

$$\frac{d\Gamma}{dt} = 0$$

# Curl of the convective acceleration term in 3D flow

$$\nabla \times (\vec{v} \cdot \nabla \vec{v}) = \nabla \times \left[ \begin{pmatrix} u'_x & u'_y & u'_z \\ v'_x & v'_y & v'_z \\ w'_x & w'_y & w'_z \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} \right] =$$

$$\boxed{\vec{\omega} = \nabla \times \begin{pmatrix} u \\ v \\ w \end{pmatrix} = \begin{pmatrix} w'_y - v'_z \\ u'_z - w'_x \\ v'_x - u'_y \end{pmatrix}}$$

$$= \begin{pmatrix} w''_{xy} - v''_{xz} & w''_{yy} - v''_{yz} & w''_{zy} - v''_{zz} \\ u''_{xz} - w''_{xx} & u''_{yz} - w''_{yx} & u''_{zz} - w''_{zx} \\ v''_{xx} - u''_{xy} & v''_{yx} - u''_{yy} & v''_{zx} - u''_{zy} \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} +$$

$$+ \begin{pmatrix} w'_x u'_y - v'_x u'_z + w'_y v'_y - v'_y v'_z + w'_z w'_y - v'_z w'_z \\ u'_x u'_z - w'_x u'_x + u'_y v'_z - w'_y v'_x + u'_z w'_z - w'_z w'_x \\ v'_x u'_x - u'_x u'_y + v'_y v'_x - u'_y v'_y + v'_z w'_x - u'_z w'_y \end{pmatrix}$$

# Vorticity transport equation for constant property Newtonian fluid

(ρ and ν are constants)

The continuity equation simplifies to: $\nabla \cdot \vec{v} = 0$

By taking the curl of the Navier-Stokes equation:

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = -\frac{1}{\rho} \nabla p + \vec{g} + \nu_0 \Delta \vec{v} \quad / \quad \nabla \times \ldots$$

???

0

we obtain

$$\frac{d\vec{\omega}}{dt} = \underbrace{\frac{\partial \vec{\omega}}{\partial t} + \boxed{\vec{v} \cdot \nabla \vec{\omega}}}_{\text{vortex transport}} = \underbrace{\nabla \times \vec{g}}_{\text{0, if g is irrotational}} + \underbrace{\nu_0 \Delta \vec{\omega}}_{\text{vortex diffusion}} + \underbrace{\boxed{\vec{\omega} \cdot \nabla \vec{v}}}_{\text{vortex stretching}} - \underbrace{\boxed{\vec{\omega} \nabla \cdot \vec{v}}}_{0.}$$

in which $\vec{\omega}$ is the vorticity vector.

$$\boxed{\vec{\omega} = \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} = \begin{pmatrix} w'_y - v'_z \\ u'_z - w'_x \\ v'_x - u'_y \end{pmatrix}}$$

$$\nabla \times (\vec{v} \cdot \nabla \vec{v}) =$$

$$\begin{pmatrix} \alpha'_x & \alpha'_y & \alpha'_z \\ \beta'_x & \beta'_y & \beta'_z \\ \gamma'_x & \gamma'_y & \gamma'_z \end{pmatrix} \begin{pmatrix} u \\ v \\ w \end{pmatrix} + \begin{pmatrix} w'_x u'_y - v'_x u'_z - u'_x w'_y + u'_x v'_z \\ u'_y v'_z - w'_y v'_x - v'_y u'_z + v'_y w'_x \\ v'_z w'_x - u'_z w'_y - w'_z v'_x + w'_z u'_y \end{pmatrix} + \begin{pmatrix} \alpha \\ \beta \\ \gamma \end{pmatrix} \begin{pmatrix} u'_x & v'_y & w'_z \end{pmatrix}$$

$$\vec{\omega} \cdot \nabla \vec{v} = \begin{pmatrix} u'_x w'_y - u'_x v'_z + \cancel{u'_y u'_z} - u'_y w'_x + u'_z v'_x - \cancel{w'_z u'_y} \\ v'_x w'_y - \cancel{v'_x v'_z} + v'_y u'_z - v'_y w'_x + \cancel{v'_z v'_x} - v'_z u'_y \\ \cancel{w'_x w'_y} - w'_x v'_z + w'_y u'_z - \cancel{w'_y w'_x} + w'_z v'_x - w'_z u'_y \end{pmatrix}$$

$$\nabla \times (\vec{v} \cdot \nabla \vec{v}) = \vec{v} \cdot \nabla \vec{\omega} - \vec{\omega} \cdot \nabla \vec{v} + \vec{\omega} \nabla \cdot \vec{v}$$

# Curl of the convective acceleration term in 2D flow

$$\vec{\omega} = \begin{pmatrix} \cancel{w'_y - v'_z} \\ \cancel{u'_z - w'_x} \\ v'_x - u'_y \end{pmatrix} \longrightarrow \boxed{\omega = v'_x - u'_y}$$

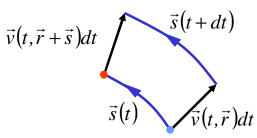
$$\nabla \times (\vec{v} \cdot \nabla \vec{v}) = \nabla \times \left[ \begin{pmatrix} u'_x & u'_y \\ v'_x & v'_y \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} \right] =$$

$$\begin{pmatrix} \underbrace{v''_{xx} - u''_{xy}}_{\omega'_x} & \underbrace{v''_{yx} - u''_{yy}}_{\omega'_y} \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \underbrace{v'_x u'_x - u'_x u'_y + v'_y v'_x - u'_y v'_y}_{\omega (u'_x + v'_y) = 0}$$

$$\nabla \times (\vec{v} \cdot \nabla \vec{v}) = \vec{v} \cdot \nabla \omega + \omega \nabla \cdot \vec{v}$$

# Vortex stretching

Evolution of a fluid line of elementary length




$$\vec{v}(t, \vec{r} + \vec{s}) - \vec{v}(t, \vec{r}) = \vec{s} \cdot \nabla \vec{v}$$

$$\frac{d\vec{s}}{dt} = \vec{s} \cdot \nabla \vec{v}$$

Vorticity transport equation for irrotational body force and zero viscosity:

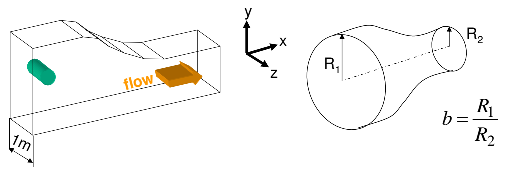
$$\frac{d\vec{\omega}}{dt} = \vec{\omega} \cdot \nabla \vec{v}$$

The direction of s is arbitrarily chosen.

Both vectors evolve according to the same transport equation, hence, in inviscid flow, the vorticity vector behaves in the same way as an infinitesimal fluid line element. (Helmholtz)

Thus, ω will grow, when the fluid line is stretched.

## Problem #1.1




Compare a 2D confuser (of slab symmetry) with an axial symmetric confuser:

- What components of the vorticity vector are non-zero?
  Use cylindrical coordinates (x,r,ϕ) in the axisymmetric case!
- In what proportion does the length of a fluid element change?
- In what proportion will the components of the vorticity change if the vortex diffusion is negligable?

To the solution

## Conclusion

The vorticity transport equation for incompressible fluids reads:

$$\frac{d\vec{\omega}}{dt} = \nabla \times \vec{g} + \nu_0 \Delta \vec{\omega} + \vec{\omega} \cdot \nabla \vec{v}$$

Origin of vorticity:
- Boundary conditions
  (wall shear)
- Non conservative forces
  (eg. Coriolis force)

Redistribution of vorticity:
- Vortex stretching
- Vortex diffusion

## Vortex diffusion

The vorticity transport equation for a 2D flow of a constant property Newtonian fluid:

$$\frac{d\omega}{dt} = \frac{\partial \omega}{\partial t} + \vec{v} \cdot \nabla \omega = \nu_0 \Delta \omega \qquad \nu_0 = \frac{\mu}{\rho} \left[\frac{m^2}{s}\right]$$

kinematical viscosity

Is in full analogy with the heat transport equation:

$$\frac{dT}{dt} = \frac{\partial T}{\partial t} + \vec{v} \cdot \nabla T = a \Delta T \qquad a = \frac{\lambda}{\rho c} \left[\frac{m^2}{s}\right]$$

heat diffusion coefficient

The kinematical viscosity can be regarded as a vorticity diffusion coefficient. These two phenomena are in full analogy.

## 2. Irrotational flows

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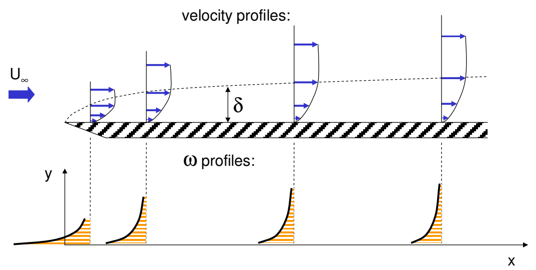
## Boundary layer over a flat plate




Vorticity is continuously produced on the wall surface due to the no-slip condition, and it is conducted into the main stream by the viscosity.

## Irrotational flows

Shape of the streamlines?
Pressure and velocity distributions?

Most analytic solutions have been developed for irrotational flows.
Lamb, H: Hydrodynamics, 1932. (First edition: 1879.)

Flows originated from a volume containing fluid at rest is an irrotational flow until the vorticity generated by walls penetrates the flow field.

„The irrotational motion of a liquid occupying a simply-connected region has less kinetic energy than any other motion consistent with the same normal motion of the boundary." (W.Thomson, 1849)

If the velocity field is rotation free: $\nabla \times \vec{v} = 0$

we can define velocity-potential function ϕ as: $\vec{v} = \nabla \phi$

(This holds for compressible flows as well.)

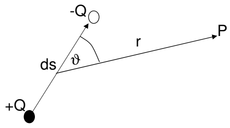
## Some application examples

- Flow close to the extraction point
- Flow around airfoils
- Darcy flow, wells
- Drinking water reservoirs

## Superposition principle

The governing equations are linear, therefore we can utilize the superposition principle.

E.g. double source (doublet).




$$ds \to 0, \quad M = Q\,ds = \text{Const.}$$

$$\phi = -\frac{M}{4\pi}\frac{\cos\vartheta}{r^2}$$

Any irrotational flow can be regarded as a result of a distribution of sources and doublets over the boundary.

The intensity distribution is still a question.
We can utilize the boundary element method …

## Calculation of the pressure field

Pressure distribution in ideal fluid (μ=0, ρ=const.) can be obtained from the Bernoulli principle:

$$p_2 - p_1 = \frac{\rho}{2}\left(v_1^2 - v_2^2\right) + \rho\, g\left(z_1 - z_2\right)$$

The equation of motion for Darcy flow:

$$\boxed{\vec{v} = -\frac{k}{\mu}\nabla\left(p + \rho g z\right)} \longrightarrow \phi = -k\frac{p + \rho g z}{\mu}$$

In which the density (ρ), the permeability (k) and the dynamic viscosity (μ) are constant values and the velocity is defined as the surface intensity of the volume flow rate:

$$Q = \int \vec{v}\, d\vec{A}$$

$$p_2 - p_1 = \frac{\mu}{k}\left(\phi_1 - \phi_2\right) + \rho\, g\left(z_1 - z_2\right)$$

## Stream function

The continuity equation of a **constant density** fluid is automatically fulfilled, if the velocity field can be derived from an existing $\underline{\Psi}$ vector potential function:

Def: $\boxed{\vec{v} = \nabla \times \vec{\psi}}$

$$\nabla \cdot \vec{v} = \nabla \cdot \nabla \times \vec{\psi} \equiv 0$$

Ψ is a scalar function in 2 spatial dimensions and called the „stream function” in 2D flow situations. Only the z component is non-zero:

$$\vec{v} = \begin{pmatrix} \frac{\partial \psi_z}{\partial y} - \cancel{\frac{\partial \psi_y}{\partial z}} \\ \cancel{\frac{\partial \psi_x}{\partial z}} - \frac{\partial \psi_z}{\partial x} \\ \cancel{\frac{\partial \psi_y}{\partial x}} - \cancel{\frac{\partial \psi_x}{\partial y}} \end{pmatrix} \longrightarrow \boxed{u = \frac{\partial \psi}{\partial y} \quad \text{and} \quad v = -\frac{\partial \psi}{\partial x}}$$

u and v are the x and y components of the velocity vector:

Ψ makes much more sense in 2D, because the definition decreases the number of unknown scalar fields.

## Velocity potential for constant density fluid flow

Continuity equation: $\nabla \cdot \vec{v} = 0$

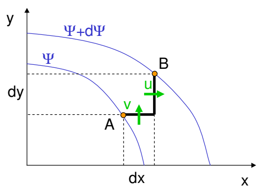
$$\nabla \cdot (\nabla \phi) = \Delta \phi = 0$$

ϕ is an harmonic function (fulfilling the Laplace equation).

An important example: velocity potential of a point source:

$$\vec{v} = \frac{Q}{4 r^2 \pi}\vec{e}_r \longrightarrow \phi = -\frac{Q}{4\pi\, r} + \text{Const.}$$

## The stream function in 2D




The total differential of the stream function:

$$d\psi = \frac{\partial \psi}{\partial x}dx + \frac{\partial \psi}{\partial y}dy$$

$$\frac{\partial \psi}{\partial x} = -v \quad \text{and} \quad \frac{\partial \psi}{\partial y} = u$$

$$d\psi = -v\,dx + u\,dy$$

Ψ expresses volume flow-rate between A and B (in a 1m wide domain):

$$Q_{A-B} = \psi_B - \psi_A$$

There is no flow through the iso-lines of Ψ, therefore these are streamlines.

The continuity in 2D: $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = \frac{\partial^2 \psi}{\partial x \partial y} - \frac{\partial^2 \psi}{\partial y \partial x} = 0$

# 2D irrotational flow of a constant density fluid

Let's suppose, that:

$$\nabla\times\vec{v}\big|_z=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}=0$$

$$\frac{\partial\Psi}{\partial x}=-v \quad \text{and} \quad \frac{\partial\Psi}{\partial y}=u$$

$$\frac{\partial^2\Psi}{\partial x^2}+\frac{\partial^2\Psi}{\partial y^2}=0$$

$\Psi$ is also a harmonic function.

# Potentials

| | $\Psi$ | $\phi$ | $w$ |
|---|---|---|---|
| Name | Stream func. | Velocity-pot. | Complex-pot. |
| Variable density flow | N.A ** | applicable | N.A |
| Rotational flow | applicable | N.A | N.A |
| 3D flow | vector | scalar | N.A |
| Definition | $\nabla\times\vec{\psi}=\vec{v}$ | $\nabla\phi=\vec{v}$ | $w=\phi+i\psi$ |

** Another definition of $\Psi$ allows compressibility.

# Complex potential (1)

Both $\Psi$ and $\phi$ are harmonic functions: $\Delta\Psi=0$ and $\Delta\phi=0$

furthermore they fulfill the Couchy-Riemann conditions:

$$u=\frac{\partial\phi}{\partial x}=\frac{\partial\psi}{\partial y} \qquad v=\frac{\partial\phi}{\partial y}=-\frac{\partial\psi}{\partial x}$$

Therefore they can be regarded as the real and imaginary parts of a differentiable complex function: $w=\phi+i\psi$ w is called complex potential.

$w=f(z)$ z is a complex number (position vector); $z=x+iy$

**Thus, any differentiable complex function corresponds to valid 2D, steady, irrotational flow of a constant density fluid.**

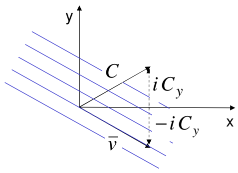
We only need to look for solutions fulfilling the boundary conditions.
We will analyze the complex potential of some primitive flow structures, then we superimpose and transform these simple solutions for obtaining solutions which fulfill more complex boundary conditions.

# Parallel flow

$w=C\,z$ C is a complex number.

$$w=(C_x+i\,C_y)(x+i\,y)=\underbrace{C_x x-C_y y}_{\phi}+i\,(\underbrace{C_y x+C_x y}_{\psi})$$

E.g: the streamline $\Psi$=0 is a straight line passing through 0,0 :




$$y=-\frac{C_y}{C_x}x$$

We can calculate the velocity:

$$\bar{v}=\frac{dw}{dz}=C=C_x+i\,C_y$$

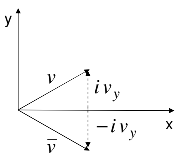
# Complex potential (2)

Velocity is a complex vector as well:

$$v=v_x+i\,v_y$$

The complex conjugate of the velocity vector can be obtained by taking the derivative of the complex potential:

$$\frac{dw}{dz}=\frac{\partial w}{\partial x}=\frac{\partial w}{\partial iy}=v_x-i\,v_y=\bar{v}$$

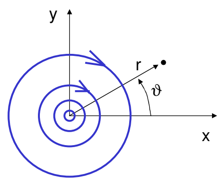



# Free vortex 1.

$w=i\,C_o\,ln\,z$ $C_0$ is a real number.

$$w=i\,C_0\,ln\left(r\,e^{i\vartheta}\right)=\underbrace{-C_0\vartheta}_{\phi}+i\,\underbrace{C_0\,ln\,r}_{\psi}$$

Streamlines are concentric circles: $\psi=C_0\,ln\,r=\text{Const.}$



# Free vortex 2.

The velocity field

$$\bar{v} = \frac{dw}{dz} = i\frac{C_0}{z} = i\frac{C_0}{r\,e^{i\vartheta}} = i\frac{C_0}{r}e^{-i\vartheta}$$

$$\bar{v} = \frac{C_0}{r}\,i\left(\cos(-\vartheta) + i\sin(-\vartheta)\right)$$

$$v = \frac{C_0}{r}\left(\sin\vartheta - i\cos\vartheta\right)$$

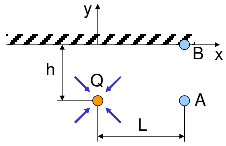
Unit vector pointing in azimuthal direction.

The velocity magnitude: $v_\vartheta = \frac{C_0}{r}$

Circulation along any curve which passes around the origo one time:

$\Gamma = 2r\pi v_\vartheta = 2r\pi\frac{C_0}{r} = 2\pi C_0$ thus: $C_0 = \frac{\Gamma}{2\pi}$

# Problem #2.2




a. Construct the complex potential for this flow! (Q, h and L are given.)
b. Determine the velocity magnitude in B!
c. What is the volume flow-rate between A and B?
d. Calculate the pressure distribution for Darcy flow of a given permeability and viscosity!

To the solution

# Problem #2.1

What is the shape of the water surface above the drain of the bath tub? Determine the drop of water level between points characterized by $r_1$ and $r_2$!

$v_z \approx 0$ the field variables depend only on r.

To the solution

# Flow around a corner

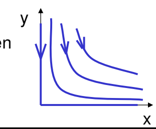
$w = \frac{C_0}{n} z^n$ $C_0$, n: real numbers, and n>0.

$$w = \frac{C_0}{n} r^n e^{in\vartheta} = \frac{C_0}{n} r^n \left(\cos n\vartheta + i\sin n\vartheta\right)$$

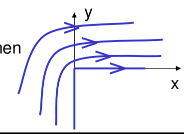
$$\psi = \frac{C_0}{n} r^n \sin n\vartheta$$

$$\psi = 0, \text{when} \quad \vartheta = 0, \frac{\pi}{n}, \frac{2\pi}{n}, \ldots$$

n=2 :
$\Psi$=0, when
0, $\pi$/2



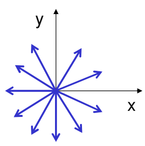

n=2/3 :
$\Psi$=0, when
0, 3$\pi$/2




# Sources and sinks

Note that, these are line sources in 3D.

$w = C_0 \ln z$ $C_0$ is a real number.




$$w = C_0 \ln\left(r\,e^{i\vartheta}\right) = \underbrace{C_0 \ln r}_{\phi} + i\underbrace{C_0\vartheta}_{\psi}$$

$$z = x + iy \longrightarrow \psi = C_0\, atg\frac{y}{x}$$

$$\bar{v} = \frac{dw}{dz} = \frac{C_0}{z} = \frac{C_0}{r}\left(\cos\vartheta - i\sin\vartheta\right)$$

$$v = \frac{C_0}{r}\left(\cos\vartheta + i\sin\vartheta\right)$$

Unit vector of radial direction.

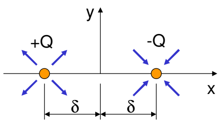
$Q\left[\frac{m^2}{s}\right] = \psi_{\vartheta=2\pi} - \psi_{\vartheta=0} = C_0\, 2\pi$ therefore: $C_0 = \frac{Q}{2\pi}$

# Problem #2.3

What is the shape of the streamlines close to a stagnation line?
y=f(x)

To the solution

## Dipoles (doublets) (1)




$$\delta \to 0,\quad Q \to \infty,\quad Q \cdot \delta = \text{const.}$$

$$w = \frac{Q}{2\pi}\left[ln(z+\delta) - ln(z-\delta)\right]$$

$$\bar{v} = \frac{Q}{2\pi}\left[\frac{1}{z+\delta} - \frac{1}{z-\delta}\right]$$

$$\bar{v} = \frac{Q}{2\pi}\,\frac{z-\delta-(z+\delta)}{z^2-\delta^2}$$

$$\boxed{w = \frac{M}{z}} \longleftarrow \quad \bar{v} = -\frac{Q\delta}{\pi}\,\frac{1}{z^2-\delta^2} = -\frac{M}{z^2}$$

## Problem #2.4

Prove that the streamlines are circular, and touching upon the x axis from the positive y direction, in the origin of the coordinate system!

To the solution

## Flow around a circular cylinder (1)

$$w = C_0 z + \frac{M}{z}$$

$$w = C_0 r\, e^{i\vartheta} + \frac{M}{r} e^{-i\vartheta} = C_0 r\,(cos\,\vartheta + i\,sin\,\vartheta) + \frac{M}{r}(cos\,\vartheta - i\,sin\,\vartheta)$$

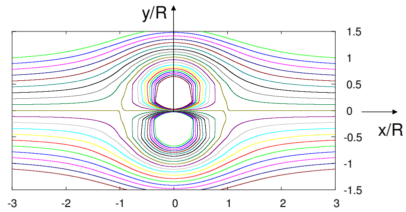
$$\Psi = \left(C_0 r - \frac{M}{r}\right) sin\,\vartheta$$

What is the equation of the streamline characterized by Ψ=0?

$\vartheta$ = 0 line and the central circle of radius R, for which: $C_0 R - \dfrac{M}{R} = 0$

$$\frac{M}{C_0} = R^2 \longrightarrow w = C_0\left(z + \frac{R^2}{z}\right)$$

## Flow around a circular cylinder (2)




$$w = C_0\left(z + \frac{R^2}{z}\right) \longrightarrow \bar{v} = C_0\left(1 - \frac{R^2}{z^2}\right)$$

$$\bar{v}\big|_{r=R} = C_0\left(1 - \frac{R^2}{R^2} e^{-2i\vartheta}\right) = C_0\left(1 - cos\,2\vartheta + i\,sin\,2\vartheta\right)$$

## Flow around a circular cylinder (3)

$$\bar{v}\big|_{r=R} = C_0\left(1 - cos\,2\vartheta + i\,sin\,2\vartheta\right)$$

$$|v|^2_{r=R} = C_0^2\left[(1 - cos\,2\vartheta)^2 + sin^2\,2\vartheta\right]$$

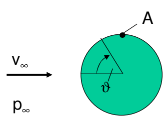
$$|v|^2_{r=R} = C_0^2\left[1 - 2\,cos\,2\vartheta + \underbrace{cos^2\,2\vartheta + sin^2\,2\vartheta}_{1}\right]$$

$$|v|^2_{r=R} = 2C_0^2\left[1 - cos^2\,2\vartheta\right]$$

$$|v|^2_{r=R} = 2C_0^2\left[\underbrace{cos^2\,\vartheta + sin^2\,\vartheta}_{1} - \left(cos^2\,\vartheta - sin^2\,\vartheta\right)\right]$$

$$|v|^2_{r=R} = 4C_0^2\,sin^2\,\vartheta \longrightarrow |v|_{r=R} = 2C_0\,|sin\,\vartheta|$$

## Problem #2.5



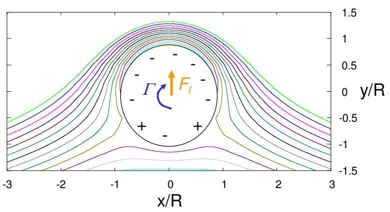

a. Calculate $v_A$ for a given $v_\infty$!
b. Determine the distribution of the pressure coefficient over the surface of the cylinder: v=f(ϑ).

To the solution

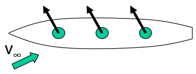
## Flettner rotor (1)

$$w = \bar{v}_\infty \left( z + \frac{R^2}{z} \right) + i \frac{\Gamma}{2\pi} \ln z$$

$$\frac{\Gamma}{v_\infty 2R\pi} = 1.6$$

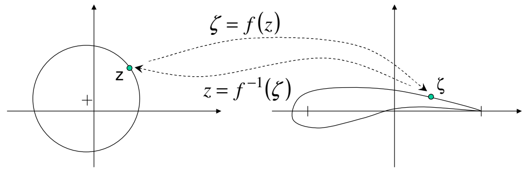



Kutta-Joukowsky theorem: $\boxed{F_l = \rho v_\infty \Gamma}$



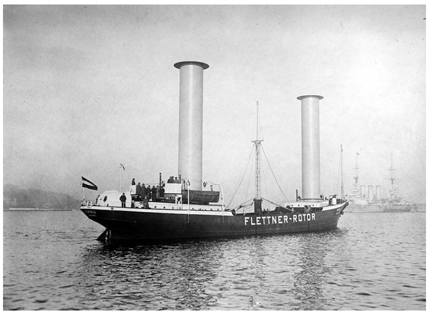
## Joukowsky transformation (1)

We transform the z space, but we keep the value of the complex potential: $w(z) = w(\zeta)$




By using the complex potential of a Flettner rotor, we can describe the flow around an airfoil.

## Flettner rotor (2)




[http://de.wikipedia.org/]

## Joukowsky transformation (2)

A complex transformation is conformal, if it does not change the far field characteristics of the function.
These transformations can be written in the form of a series:

$$\zeta = f(z) = z + \frac{a_1}{z} + \frac{a_2}{z^2} + \frac{a_3}{z^3} + \ldots$$

in which $a_1, a_2, a_3, \ldots$ are complex numbers.

The simplest possible case is the Joukowsky transformation:

$$\boxed{\zeta = z + \frac{a_{10}}{z}}$$

in which $a_{10}$ is a **real** number.

## Problem #2.6

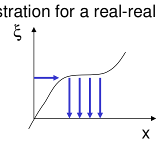
What circulation intensity is necessary for shifting the stagnation point by $\vartheta_0$ angle?

To the solution

## Singular points (1)

In those points where the derivative of the transformation expression is zero, the inverse transformation is not single valued.

A simplified illustration for a real-real transformation:




$$f'(x) = \frac{d\xi}{dx} = 0$$

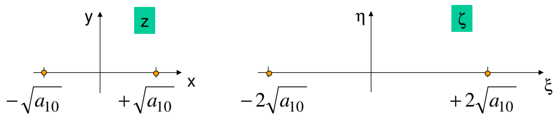
$$\zeta = z + \frac{a_{10}}{z} \longrightarrow \frac{d\zeta}{dz} = 1 - \frac{a_{10}}{z^2} = 0$$

The singular points are on the real axis in: $z = \pm\sqrt{a_{10}}$

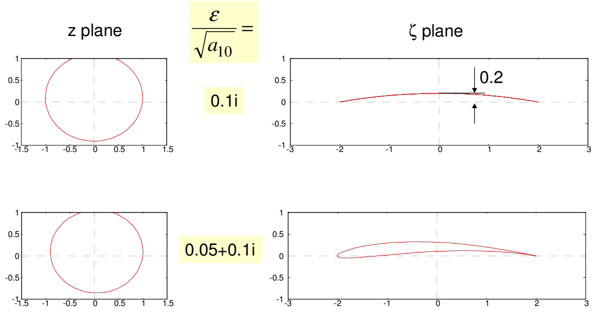
# Singular points (2)

The transformed images of the singular points:

$$\zeta = \pm\sqrt{a_{10}} + \frac{a_{10}}{\pm\sqrt{a_{10}}} = \pm 2\sqrt{a_{10}}$$




# Joukowsky profiles (2)



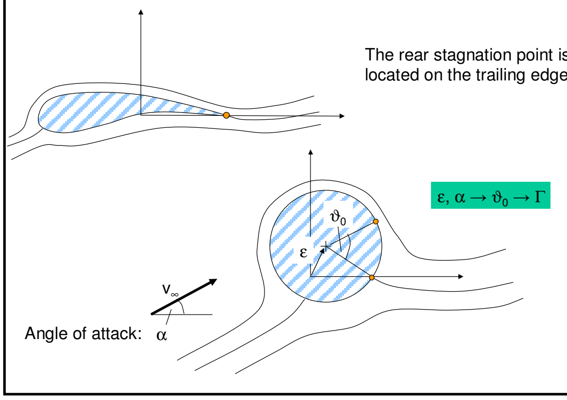

# Problem #2.7

Please, specify the equation of a circle around the complex point ε, passing through the real point $\sqrt{a_{10}}$.

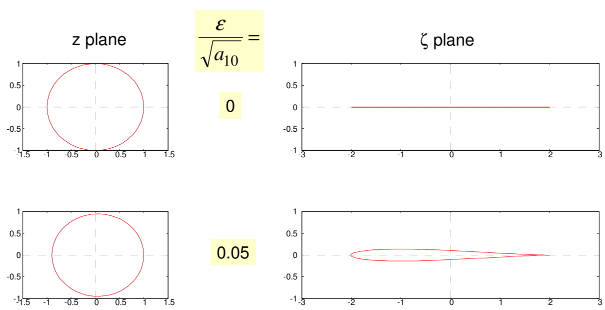
To the solution

# Kutta condition

The rear stagnation point is located on the trailing edge.

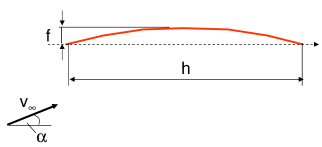



# Joukowsky profiles (1)




# Problem #2.8

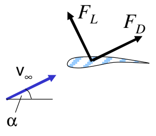
Estimate the lift coefficient ($c_l$) of an arched plate!
α and f/h can be regarded as given values, with both being small.




$$c_L = \frac{F_L}{\frac{\rho}{2} v_\infty^2 A}$$

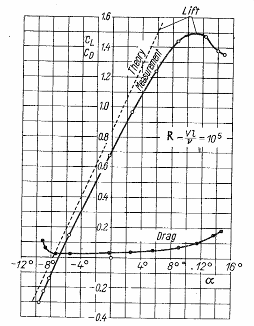
To the solution

## Comparison with measured data




$$c_L = \frac{F_L}{\frac{\rho}{2} v_\infty^2 A}$$

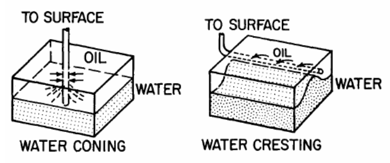
$$c_D = \frac{F_D}{\frac{\rho}{2} v_\infty^2 A}$$

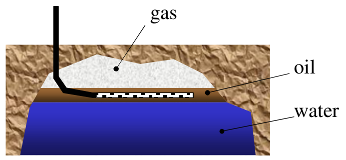



Lift and drag coefficients for a Joukowsky profile

[Schlichting 1.11]

## Oil wells



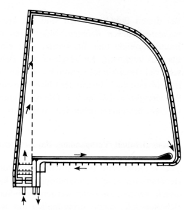

# Application examples

Department of Fluid Mechanics, BME



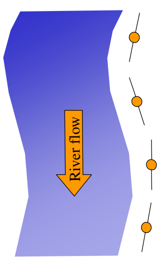
## Guber József Water Reservoir Budapest

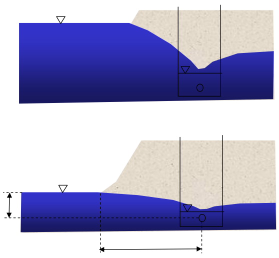
**The plans of a state of the art water reservoir operating in Munich was adapted by the Budapest water company in 1970.**



**2 piano shaped reservoirs 40.000 m³ each.**

## Onshore water wells



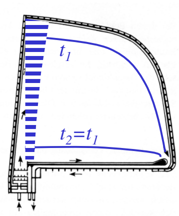



The critical flow rate depends on:
1. River level
2. The sustainable permittivity of the infiltration surface
3. Critical velocity around the pipes for avoiding damage in the porous medium.

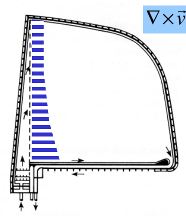
## Operating modes

**Munich**
Continuous flow.
The total amount of water produced by the supplier passes through the reservoir.

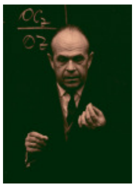
**Budapest**
Used for network pressure stabilization.
Loaded by night, and unloaded during the peak consumption hours.




$\nabla \times \vec{v} = 0$ !

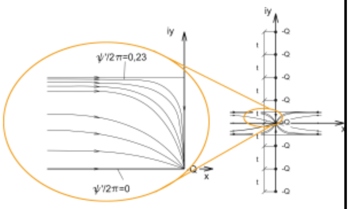


## Professor József Gruber (1915-1972)



Head of Department at the Dept. Of Fluid Mechanics, BME between 1950 and 1972

Proposed the idea of irrotational flow as a design target. He also suggested a method for finding an analitical solution for the irrotational flow field.

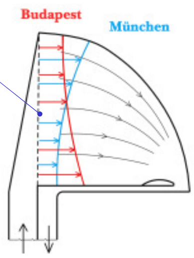



Infinite series of sinks



## Laboratory experiments

**Variable inlet comb**




- Experimental setup
- Inlet comb with uniform perforation
- The Munich case
- The Budapest case