3. Boundary layers

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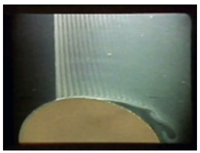
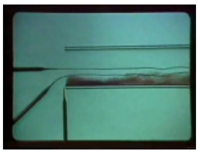
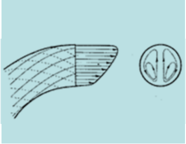
Boundary layer related phenomena


[Shapiro]


[Shapiro]


[Schlichting 20.25]

Separation:
- formation of free shear layer,
- strong modification of the surface pressure distribution (increased head loss),
- production and also reduction of the lift force acting on wings.

Turbulence:
- irregular velocity fluctuations
- increased BL thickness
- increased transport coefficient (local heat transfer coef. skin friction)
- increased resistance against separation

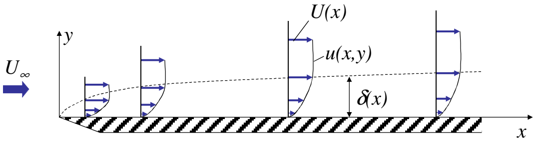
Secondary flow:
- by-passing fluid from high to low surface pressure zones,
- creation of vorticity parallel to the main stream
- increased mixing, drift motion of sediments and buoyant particles

Displacement: Virtually increases the thickness of a plate or an airfoil.

The boundary layer concept

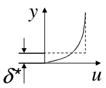
If the fluid viscosity is very small, then surface friction can effect the flow only in the immediate vicinity of the wall, in a layer of δ thickness.

We will discuss only steady 2D cases: $\vec{v} = u\,\vec{e}_x + v\,\vec{e}_y$



Boundary layer thickness

Definitions:



$\delta:$ $u(\delta) = 0.99U$ along an x=const. line.

$\delta^*:$ $U\delta^* = \int_0^\infty (U - u(y))dy$ displacement thickness

For flat plates of 0 inclination: $\delta \cong 3.26\delta^*$

We can estimate δ assuming a balance between viscous and inertial forces at the edge of the boundary layer (y=δ). If ν_0 is a constant value:

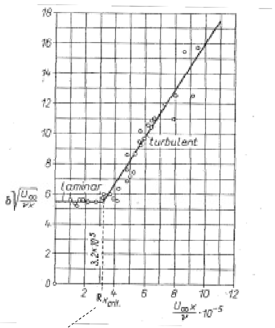
$$u\frac{\partial u}{\partial x} \cong \nu_0 \frac{\partial^2 u}{\partial y^2}$$

$$U_\infty \frac{U_\infty}{x} \sim \nu_0 \frac{U_\infty}{\delta^2}$$

$$\frac{\delta}{x} \sim \underbrace{\sqrt{\frac{\nu_0}{U_\infty x}}}_{Re_x^{-0.5}}$$

Evolution of δ on a flat plate of 0 inclination

[Schlichting 2.16]



$Re_{x,crit} = 3.2\times10^5$

Two alternative definitions of the Reynolds number:

$$Re_x = \frac{U_\infty x}{\nu_0}$$

$$Re_\delta = \frac{U_\infty \delta}{\nu_0}$$

For laminar boundary layer:

$$\frac{Re_\delta}{Re_x} = \frac{\delta}{x} = 5.64\,Re_x^{-0.5}$$

$$Re_\delta = 5.64\,Re_x^{0.5}$$

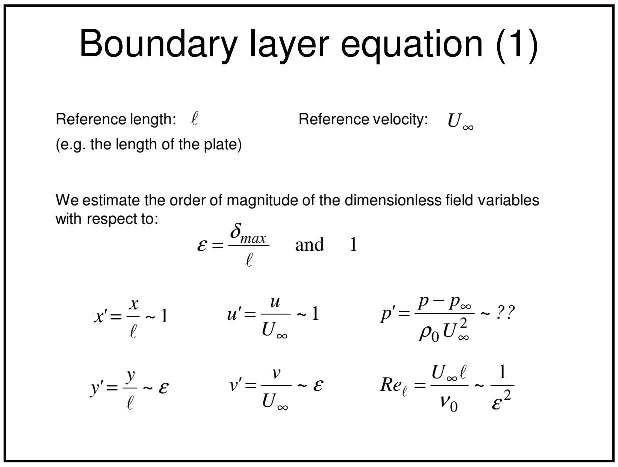
Problem #3.1

A) Compare the critical value of Re_δ (corresponding to laminar-turbulent transition) for a flat plate and in a circular pipe by assuming:

$$\delta \cong \frac{D}{2}$$

B) What is the dimensionless transition length x_{crit}/D at the critical value of Re_D ?

To the solution



Boundary layer equation (1)

Reference length: ℓ (e.g. the length of the plate)

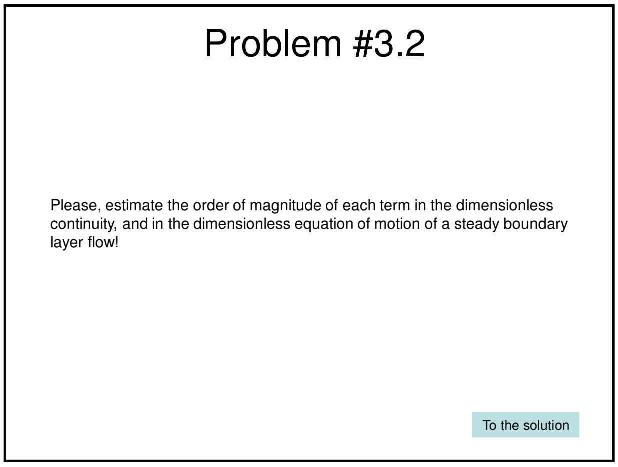
Reference velocity: U_∞

We estimate the order of magnitude of the dimensionless field variables with respect to:

$$\varepsilon = \frac{\delta_{max}}{\ell} \quad \text{and} \quad 1$$

$$x' = \frac{x}{\ell} \sim 1 \qquad u' = \frac{u}{U_\infty} \sim 1 \qquad p' = \frac{p - p_\infty}{\rho_0 U_\infty^2} \sim ??$$

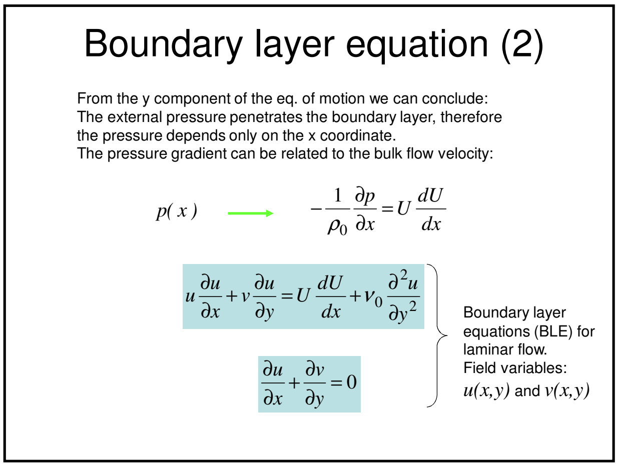
$$y' = \frac{y}{\ell} \sim \varepsilon \qquad v' = \frac{v}{U_\infty} \sim \varepsilon \qquad Re_\ell = \frac{U_\infty \ell}{\nu_0} \sim \frac{1}{\varepsilon^2}$$



Problem #3.2

Please, estimate the order of magnitude of each term in the dimensionless continuity, and in the dimensionless equation of motion of a steady boundary layer flow!

To the solution



Boundary layer equation (2)

From the y component of the eq. of motion we can conclude:
The external pressure penetrates the boundary layer, therefore the pressure depends only on the x coordinate.
The pressure gradient can be related to the bulk flow velocity:

$$p(x) \xrightarrow{\qquad} -\frac{1}{\rho_0}\frac{\partial p}{\partial x} = U\frac{dU}{dx}$$

$$u\frac{\partial u}{\partial x} + v\frac{\partial u}{\partial y} = U\frac{dU}{dx} + \nu_0\frac{\partial^2 u}{\partial y^2}$$

$$\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} = 0$$

Boundary layer equations (BLE) for laminar flow.
Field variables: $u(x,y)$ and $v(x,y)$

Self-similarity of the laminar boundary layer

$$\frac{\partial u'}{\partial x'}+\frac{\partial v'}{\partial y'}=0 \qquad u'\frac{\partial u'}{\partial x'}+v'\frac{\partial u'}{\partial y'}=U'\frac{dU'}{dx'}+\frac{1}{Re_\ell}\frac{\partial^2 u'}{\partial y'^2}$$

We perform another scaling:

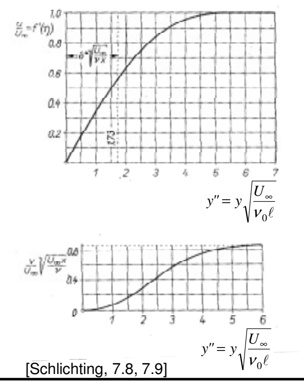
$y''=y'\sqrt{Re_\ell}=\frac{y}{\ell}\sqrt{\frac{U_\infty \ell}{\nu_0}}$ and $v''=v'\sqrt{Re_\ell}=\frac{v}{U_\infty}\sqrt{\frac{U_\infty \ell}{\nu_0}}$

the dimensionless BLE reads:

$$\frac{\partial u'}{\partial x'}+\frac{\partial v''}{\partial y''}=0 \qquad u'\frac{\partial u'}{\partial x'}+v''\frac{\partial u'}{\partial y''}=U'\frac{dU'}{dx'}+\frac{\partial^2 u'}{\partial y''^2}$$

The solutions of this form are independent from Re_l : $u'(x', y'')$

Flat plate of 0 inclination



[Schlichting, 7.8, 7.9]

Solved by Blasius (1908).

$\delta :\quad y''=5.64$

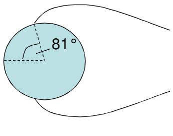
$\delta^* :\quad y''=1.73$

$$\delta=3.26\,\delta^*$$

Due to the self-similarity, these profiles are independent from Re_x.

Flow past a cylinder

The position of the separation point must be independent from the Reynolds number. (As long as the external flow is independent from Re.)



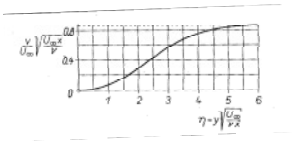
$x'=\frac{x}{\ell}\propto \text{angle}\quad 0\le x'\le\frac{\ell\pi}{2}$ indep. from Re_l.

The external flow Is irrotational, $U'\frac{dU'}{dx'}$ indep. from Re_l.

Condition for separation: $\left.\frac{\partial u'}{\partial y''}\right|_{y''=0}=0$ indep. from Re_l.

Problem #3.3

Please, calculate the displacement velocity v(x,δ)
(y velocity profile at the edge of the boundary layer)
over a flat plate of zero inclination for given l, Re_l and U_∞ .



To the solution

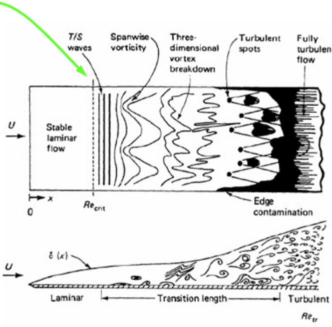
Origin of turbulence

Instability of the laminar boundary layer: exponential growth of the amplitude of Tollmien-Schlichting waves.

Effects helping the transition:

1. **Natural transition**
   The initial disturbances are generated by the uneven surface. Amplification rate depends on dp/dx.
2. **Bypass transition**
   The transition is boosted by the turbulence of the main flow.
3. **Separation induces transition**
   Laminar separation creates an inflexion in the u(y) profile which is unstable.
4. **Cross-flow transition**
   Instability caused by a cross flow (w velocity component) e.g. past swept wings or rotating bodies.

Top view:



[White: Viscous Fluid Flow, 1991]

The method of small perturbations (1)

The flow quantities are decomposed: $u = \bar{u} + \tilde{u}$ $\quad v = \bar{v} + \tilde{v}$ $\quad p = \bar{p} + \tilde{p}$

The mean flow is a 2D quasi-steady boundary layer flow: $\bar{u}(y),\ \ \bar{v} \approx 0,\ \ \bar{p}(x)$

Small perturbations (2D, time dependent): $\tilde{u}(x,y,t),\ \ \tilde{v}(x,y,t),\ \ \tilde{p}(x,y,t)$

Quadratic terms of the perturbation velocity are neglected.

The perturbed flow:

$$\frac{\partial \tilde{u}}{\partial x} + \frac{\partial \tilde{v}}{\partial y} = 0$$

$$\frac{\partial \tilde{u}}{\partial t} + \bar{u}\frac{\partial \tilde{u}}{\partial x} + \tilde{v}\frac{\partial \bar{u}}{\partial y} = -\underline{\frac{\partial \bar{p}}{\partial x}} - \frac{\partial \tilde{p}}{\partial x} + \nu_0\left(\frac{\partial^2 \tilde{u}}{\partial x^2} + \underline{\frac{\partial^2 \bar{u}}{\partial y^2}} + \frac{\partial^2 \tilde{u}}{\partial y^2}\right)$$

$$\frac{\partial \tilde{v}}{\partial t} + \bar{u}\frac{\partial \tilde{v}}{\partial x} = -\frac{\partial \tilde{p}}{\partial y} + \nu_0\left(\frac{\partial^2 \tilde{v}}{\partial x^2} + \frac{\partial^2 \tilde{v}}{\partial y^2}\right)$$

The mean flow:

$$0 = -\frac{\partial \bar{p}}{\partial x} + \nu_0\frac{\partial^2 \bar{u}}{\partial y^2}$$

The method of small perturbations (2)

$$\frac{\partial \tilde{u}}{\partial x}+\frac{\partial \tilde{v}}{\partial y}=0$$

$$\frac{\partial \tilde{u}}{\partial t}+\bar{u}\frac{\partial \tilde{u}}{\partial x}+\tilde{v}\frac{\partial \bar{u}}{\partial y}=-\frac{\partial \tilde{p}}{\partial x}+\nu_0\left(\frac{\partial^2 \tilde{u}}{\partial x^2}+\frac{\partial^2 \tilde{u}}{\partial y^2}\right)$$

$$\frac{\partial \tilde{v}}{\partial t}+\bar{u}\frac{\partial \tilde{v}}{\partial x}=-\frac{\partial \tilde{p}}{\partial y}+\nu_0\left(\frac{\partial^2 \tilde{v}}{\partial x^2}+\frac{\partial^2 \tilde{v}}{\partial y^2}\right)$$

By introducing the stream function Ψ, for which $\tilde{u}=\frac{\partial \psi}{\partial y}$ and $\tilde{v}=-\frac{\partial \psi}{\partial x}$

The continuity equation is automatically fulfilled.

Furthermore, we can eliminate the pressure by taking the curl of the equation of motion. The result would be a forth order PDE for Ψ …

Tollmien-Schlichting waves

We are looking for the solution in wave form: $\psi(x,y,t)=f(y)\,e^{i(\alpha x-\beta t)}$

Note that, $f(y)$ is complex, but physical meaning is only given for the real part.

$\alpha=\frac{2\pi}{\lambda}$ α is a real quantity;

$\beta=\beta_r+i\,\beta_i$ β_r : angular frequency, β_i : amplification factor.

$$c=\frac{\beta}{\alpha}=c_r+i\,c_i$$

$$\tilde{u}=\frac{\partial \psi}{\partial y}=f'(y)e^{i(\alpha x-\beta t)}$$

$$\tilde{v}=-\frac{\partial \psi}{\partial x}=-i\,\alpha\,f(y)e^{i(\alpha x-\beta t)}$$

Problem #3.4

Please, calculate the vorticity of the perturbation velocity field for Tollmien-Schlichting waves!

To the solution

Stability equation (1)

After substitution and elimination of the pressure, we obtain a 4-th order ordinary differential equation for $f(y)$:

$$(\bar{u}-c)\left(f''-\alpha^2 f\right)-\bar{u}''f=-\frac{i\nu_0}{\alpha}\left(f''''-2\alpha^2 f''+\alpha^4 f\right)$$

We can assume the following boundary conditions:

$$y=0: \quad \tilde{u}=\tilde{v}=0 \longrightarrow f=f'=0$$

$$y\to\infty: \quad \tilde{u}=\tilde{v}=0 \longrightarrow f=f'=0$$

Dimensionless quantities:

$$\frac{y}{\delta^*},\ \frac{\tilde{u}}{U_\infty},\ \frac{\tilde{v}}{U_\infty},\ \frac{U_\infty\delta^*}{\nu_0},\ \alpha\delta^*,\ \frac{\beta_i\delta^*}{U_\infty}$$

Re_{δ^*} wave number amplification factor

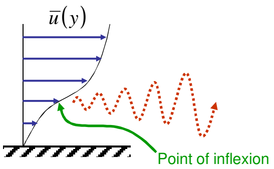
Stability equation (2)

$$(\bar{u}-c)\left(f''-\alpha^2 f\right)-\bar{u}''f=-\frac{i\nu_0}{\alpha}\left(f''''-2\alpha^2 f''+\alpha^4 f\right)$$

Stability of BL depends on

$\bar{u}(y)$

Re_{δ^*}

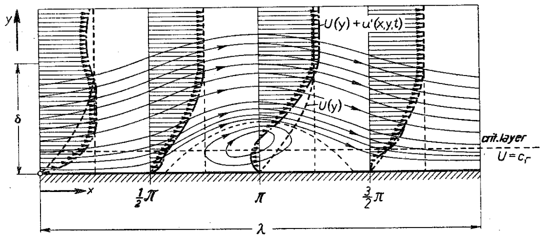


„Velocity profiles with a point of inflexion are unstable." /Rayleigh – Tollmien theorem/

Eg. the Blasius profile is unstable above a certain critical Reynolds number.

Flow pattern

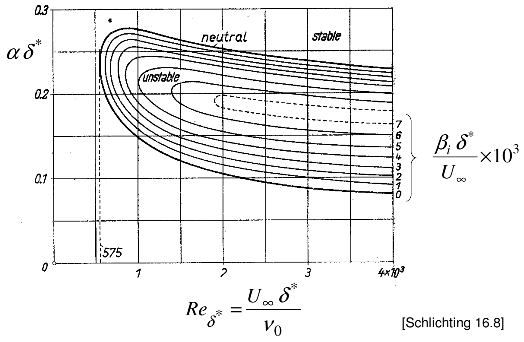
… for a neutral (*β_i=0*) disturbance in a given mean BL profile at given Re_{δ^*}.



[Schlichting 16.9]

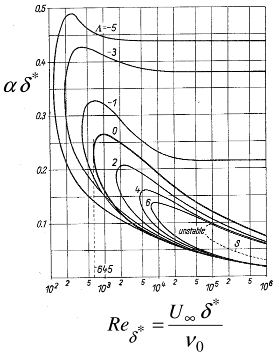
Amplification of the disturbances

(Flat plate with 0 pressure gradient)



[Schlichting 16.8]

Effect of the pressure gradient



$$\Lambda = \frac{\delta^2}{\nu_0}\frac{dU}{dx}$$

$\Lambda<0$: diffuser
$\Lambda>0$: confuser

The pressure gradient is linked with the velocity gradient of the external flow:

$$\rho_0\,U\frac{dU}{dx} = -\frac{dp}{dx}$$

Adverse pressure gradient

⇩

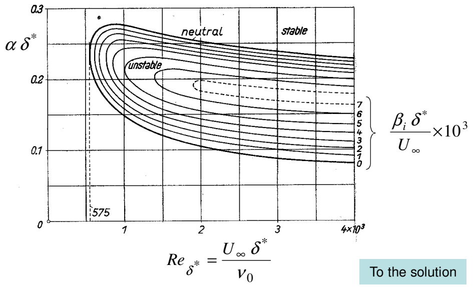
Formation of an inflexion point on the mean velocity profile

⇩

High amplification factor for a wide range value of α.

Problem #3.5

Please, calculate the displacement thickness and the wavelength of highest amplification factor for a flat plate of zero inclination at Re_x=200000, x=0.1 m. (This is roughly a speed of 108 km/h in standard atmosphere.)



To the solution

Averaging

Turbulent motion is **irregular**: you will possibly measure N different values at the same flow time (time elapsed from the start of the experiment) and spatial coordinates if you repeat the experiment N times.

The expected values of the measured quantities are denoted by over-bar and regarded as mean flow quantities. Eg:

$$\overline{u} = \lim_{N\to\infty}\left(\frac{1}{N}\sum_{i=1}^{N} u_i\right)$$

Mean values in a quasi-steady flow can be approximated by the temporal average of a measured signal recorded during a sufficiently long time interval T:

$$\overline{u} \cong \langle u\rangle = \frac{1}{T}\int_{t-T/2}^{t-T/2} u(t)\,dt$$

Effect of turbulence on mean flow: Reynolds averaging

We decompose the instantaneous flow quantities to mean values and turbulent fluctuations:

$$u=\overline{u}+u' \qquad v=\overline{v}+v' \qquad w=\overline{w}+w' \qquad p=\overline{p}+p'$$

The mean values of all fluctuating quantities are zero and the average values are approximately zero as well:

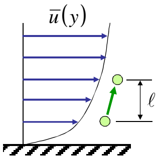
$$\overline{u'}=0 \qquad \text{and} \qquad \langle u'\rangle \cong 0$$

By taking the average of the Navier-Stokes equation for the instantaneous flow field, for incompressible flow we obtain:

$$\underbrace{\rho_0\frac{\partial\langle\vec{v}\rangle}{\partial t}+\rho_0\langle\vec{v}\rangle\cdot\nabla\langle\vec{v}\rangle=-\nabla\langle p\rangle+\rho_0\vec{g}+\mu_0\Delta\langle\vec{v}\rangle}_{\textbf{NS equation for the mean flow}}-\underbrace{\rho_0\langle\vec{v}'\cdot\nabla\vec{v}'\rangle}_{\textbf{Reynolds stresses}}$$

Must be given in order to close the set of equations

Prandtl's mixing length model



1.) The fluctuation magnitude caused by a fluid parcel which is displaced over a distance l can be expressed as:

$$u'=\ell\frac{d\overline{u}}{dy}$$

in which the mixing length l can be properly approximated as a function of mean flow characteristics and geometrical parameters.

2.) All components of the fluctuating velocity are approximately the same:

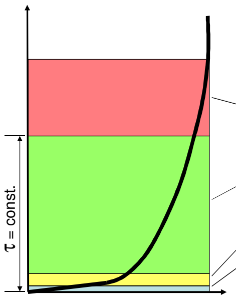
$$u'\cong v'$$

On the basis of the above assumptions we can calculate the components of the Reynolds stress tensor. Eg:

$$\rho_0\langle u'v'\rangle=\rho_0\ell^2\left|\frac{\partial\overline{u}}{\partial y}\right|\frac{\partial\overline{u}}{\partial y}=\rho_0\,\nu_t\,\frac{\partial\overline{u}}{\partial y}$$

turbulent viscosity (not a constant)

Structure of the turbulent boundary layer



4. Outer layer → deceleration
3. Fully turbulent layer → $\boxed{\overline{u} \propto ln(y)+C}$
2. Buffer layer
1. Viscous sub-layer → $\boxed{\overline{u} \propto y}$

Velocity profile

Viscous sub-layer

$$\tau = \rho_0 \nu_0 \frac{d\overline{u}}{dy} = \tau_w$$

$$\boxed{u^* = \sqrt{\frac{\tau_w}{\rho_0}}}$$

$$(u^*)^2 = \nu_0 \frac{d\overline{u}}{dy}$$

$$\boxed{\frac{\overline{u}}{u^*} = \frac{u^* y}{\nu_0} = y^+}$$

0<y₊<5 (..10) → $0<y_+<5$ (..10)

Fully turbulent layer

$$\tau = \rho_0 \ell^2 \left(\frac{d\overline{u}}{dy}\right)^2 \cong \tau_w$$

$$\boxed{\ell = \kappa\, y}$$

Von Kármán constant: $\boxed{\kappa = 0.4}$

$$(u^*)^2 = \kappa^2 y^2 \left(\frac{d\overline{u}}{dy}\right)^2$$

$$\frac{d\overline{u}}{u^*} = \frac{1}{\kappa}\frac{dy}{y}$$

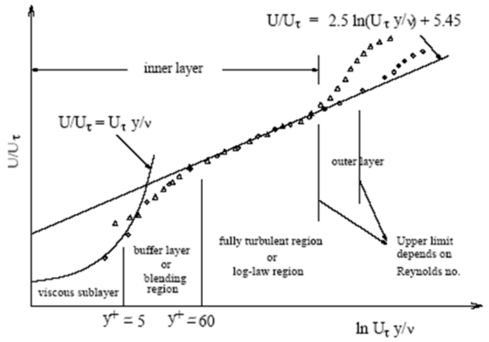
$$\boxed{\frac{\overline{u}}{u^*} = \frac{1}{\kappa} ln\left(\underbrace{\frac{u^* y}{\nu_0}}_{y^+}\right) + C}$$

For smooth plate: $\boxed{C = 5.45}$

(30..) 60<y_+<300 ?

(C is roughness dependent.)

Velocity profile



[ANSYS-FLUENT manual]

Problem #3.6

Determine the turbulent viscosity ratio (ν_t / ν_0) in the logarithmic layer for a given value of y+!

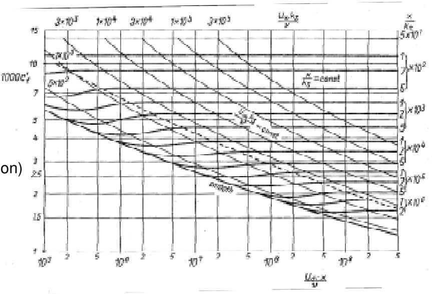
To the solution

The effect of the surface roughness

Skin friction coefficient for a flat plate

$$c'_f = \frac{\tau_0}{\frac{\rho_0}{2} U_\infty^2}$$

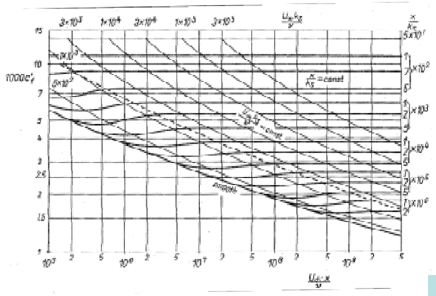
k_s: sand roughness (height of a protrusion)



$$c'_f = \left(2.87 + 1.58 log \frac{x}{k_s}\right)^{-2.5}$$ For $10^2 < \frac{x}{k_s} < 10^6$ [Schlichting 21.10]

Problem #3.7

Determine the maximum magnitude of sand roughness for which a flat plate can be regarded as hydraulically smooth. The free stream velocity and the kinematical viscosity are given: $U_\infty = 15\,\mathrm{m/s}, \quad \nu_0 = 1.5 \times 10^{-5}\,\mathrm{m^2 s^{-1}}$

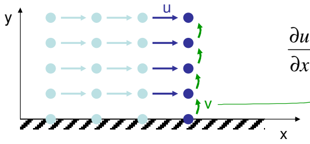


To the solution

Numerical integration of the BLE

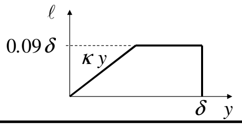
Parabolic PDE for u(x,y) and v(x,y).

$$u\boxed{\frac{\partial u}{\partial x}}+v\frac{\partial u}{\partial y}=U\frac{dU}{dx}+\frac{\partial}{\partial y}\left((\nu_t+\nu_0)\frac{\partial u}{\partial y}\right)$$



$$\frac{\partial u}{\partial x}+\boxed{\frac{\partial v}{\partial y}}=0$$

l is overestimated by the expression κy in the **outer layer,** therefore l **must be limited**.
Escudier correlation:



Solution of heat and mass transfer problems

When u and v are already known we can calculate T (temperature) and c (concentration) fields.

$$u\boxed{\frac{\partial T}{\partial x}}+v\frac{\partial T}{\partial y}=\frac{\partial}{\partial y}\left((a_t+a_0)\frac{\partial T}{\partial y}\right)$$

$$u\boxed{\frac{\partial c}{\partial x}}+v\frac{\partial c}{\partial y}=\frac{\partial}{\partial y}\left((D_t+D_0)\frac{\partial c}{\partial y}\right)$$

heat cond. coeff.

Heat diffusivity coefficient [m^{2}s^{-1}]: $a=\dfrac{\lambda}{\rho_0\, c_p}$

specific heat at const pressure

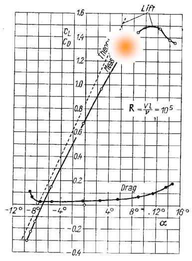
Transport coefficients are calculated from ν_t:

$Pr_t=\dfrac{\nu_t}{a_t}$ ← Turb. **Prandtl number** (given, empirical val.)

$Sc_t=\dfrac{\nu_t}{D_t}$ ← Turb. **Schmidt number** (given, empirical val.)

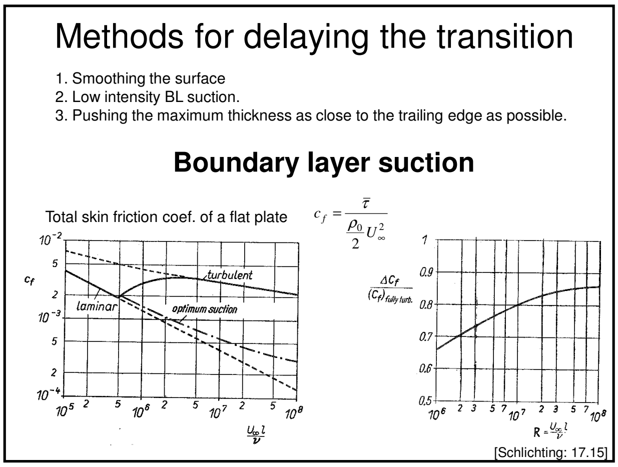
Performance of airfoils

$$F_L=c_L\frac{\rho}{2}v_\infty^2A$$



Requirements

High lift at low speed	Low drag at high speed
For low speed takeoff and landing ability.	For minimum fuel consumption.
⇩	⇩
Avoiding BL separation	Delaying BL transition



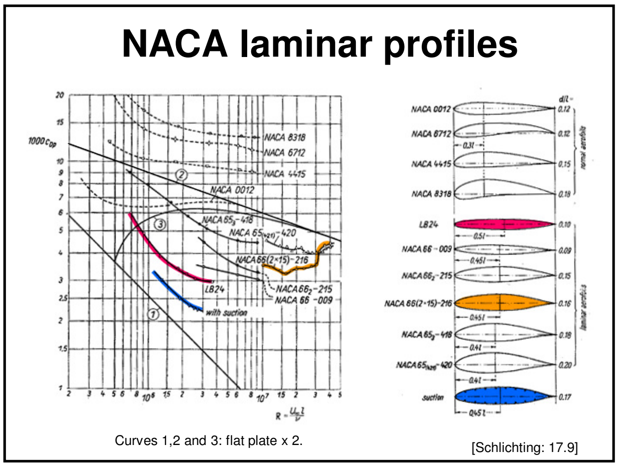
Methods for delaying the transition

1. Smoothing the surface
2. Low intensity BL suction.
3. Pushing the maximum thickness as close to the trailing edge as possible.

Boundary layer suction

Total skin friction coef. of a flat plate $c_f = \frac{\bar{\tau}}{\frac{\rho_0}{2} U_\infty^2}$

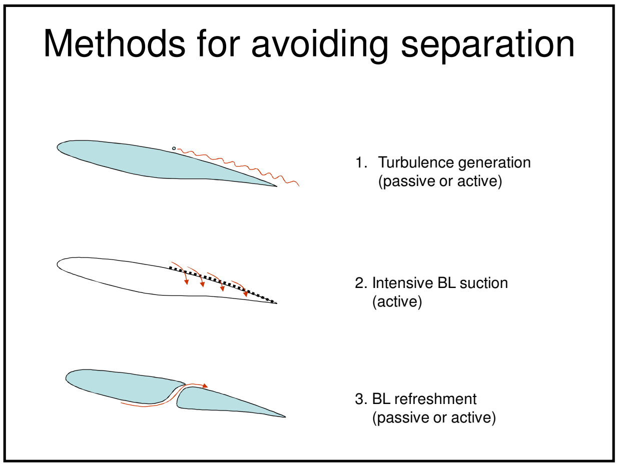
[Schlichting: 17.15]



NACA laminar profiles

Curves 1,2 and 3: flat plate x 2.

[Schlichting: 17.9]



Methods for avoiding separation

1. Turbulence generation (passive or active)
2. Intensive BL suction (active)
3. BL refreshment (passive or active)