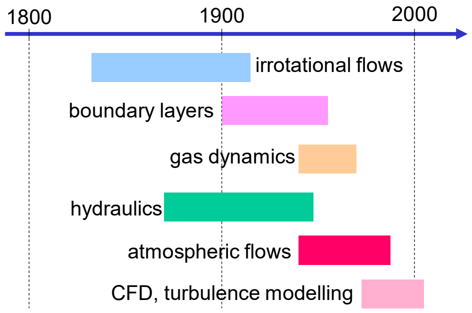
Advanced Fluid Mechanics

BME GEÁT MW01

Dr. Gergely Kristóf
Department of Fluid Mechanics, BME
February 2019.

Timeline



Foreseeable program

Course week		Topics
🟦	1.	Overview of the fundaments of fluid mechanics. Vorticity transport equation.
🟦	2.	Irrotational flows, Darcy flow. Wells.
🟦	3.	Joukowsky transformation.
🟪	4.	Boundary layers. Similarity solutions for laminar and turbulent boundary layers.
🟪	5.	Transition. Turbulent boundary leyers. BL control.
🟧	6.	Fundaments of gasdynamics. Wave phenomena. Izentropic flow
🟧	7.	Normal shock waves
🟧	8.	Oblique shock waves, wave reflection. Prandtl-Meyer expansion, Supersonic jets.
🟩	9.	Pipe networks
🟩	10.	Pipe transients
🟥	11.	Atmospheric flows
	12.	--- (Faculty Day of Profession)
	13.	--- (May 1)
🟪	14.	Martijn Schoot: Application of computational fluid dynamics (CFD) in pump design.

References

Lecture handouts:
http://www.ara.bme.hu/oktatas/tantargy/NEPTUN/BMEGEATMW01/2018-2019-II/ea_lecture/

For further reading:

1) Lamb H: Hydrodynamics, 1932.
2) Schlichting H: Boundary Layer Theory, 1955.
3) Shapiro A. H: The Dynamics and Thermodynamics of Compressible Fluid Flow, 1953.
4) Streeter V. L, Wylie E. B: Fluid Mechanics, McGraw-Hill, 1975.
5) Ferziger J. H, Peric M: Computational Methods for Fluid Dynamics, Springer, ISBN 3-540-42074-6, 2002.

1. Vorticity transport

Dr. Gergely Kristóf
Dept. of Fluid Mechanics, BME
February, 2019.

Acceleration of a fluid parcel

Motion of clouds: rudder, satelite

Velocity components: $\vec{v}(t,\vec{r})=u(t,x,y,z)\vec{i}+v(t,x,y,z)\vec{j}+w(t,x,y,z)\vec{k}$

$$du=\frac{\partial u}{\partial t}dt+\frac{\partial u}{\partial x}dx+\frac{\partial u}{\partial y}dy+\frac{\partial u}{\partial z}dz$$

For a fluid parcel: $\frac{dx}{dt}=u,\quad \frac{dy}{dt}=v,\quad \frac{dz}{dt}=w$

Substantial derivatives

$$\left.\begin{aligned}\frac{du}{dt}&=\frac{\partial u}{\partial t}+\vec{v}\cdot\nabla u\\ \frac{dv}{dt}&=\frac{\partial v}{\partial t}+\vec{v}\cdot\nabla v\\ \frac{dw}{dt}&=\frac{\partial w}{\partial t}+\vec{v}\cdot\nabla w\end{aligned}\right\}\longrightarrow \frac{d\vec{v}}{dt}=\underbrace{\frac{\partial \vec{v}}{\partial t}}_{\text{local acceleration}}+\underbrace{\vec{v}\cdot\overbrace{\nabla\vec{v}}^{\text{velocity gradient tensor}}}_{\text{convective acceleration}}$$

Navier-Stokes equation

$$\rho = const, \quad \nu = const$$

$$\frac{d\vec{v}}{dt} = \frac{\partial \vec{v}}{\partial t} + \vec{v} \cdot \nabla \vec{v} = \underbrace{-\frac{1}{\rho}\nabla p}_{\text{pressure force}} + \underbrace{\vec{g}}_{\text{body-force}} + \underbrace{\nu \Delta \vec{v}}_{\text{shear}} \qquad \left[\frac{N}{kg}\right]$$

Continuity

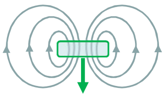
$$\underbrace{\frac{\partial}{\partial t}\int_V \rho \, dV}_{\text{accumulated mass}} + \underbrace{\oint_A \rho \underline{v} \cdot dA}_{\text{mass outflux}} = 0$$

which yields: $\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \underline{v}) = 0$

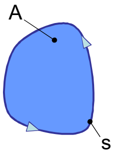
If ρ=const: $\nabla \cdot \underline{v} = 0$ thus $\frac{\partial u}{\partial x} + \frac{\partial v}{\partial y} + \frac{\partial w}{\partial z} = 0$

Stream tubes can form closed tubes or have both ends on boundaries.

Eg. absolute streamlines around
a moving object:



Vortices



Circulation:

$$\Gamma = \oint_s \vec{v} \cdot d\vec{s}$$

Vorticity:

$$\vec{\omega} = \nabla \times \vec{v} = \begin{pmatrix} w'_y - v'_z \\ u'_z - w'_x \\ v'_x - u'_y \end{pmatrix}$$

From the Stokes theorem:
On a singly-connected
surface.

$$\Gamma = \int_A \vec{\omega} \cdot d\vec{A}$$

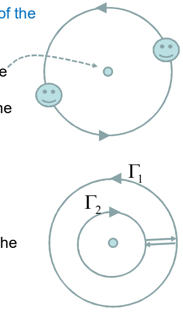
Can we have circulation without vorticity?

Free vortex

Vorticity is 2 times the angular velocity of the fluid parcel.

You must exclude this central part, otherwise then the domain is singly connected and circulation is not possible.

Vorticity is concentrated in the center.



$$\omega_{\perp}=0$$

$$\Gamma_1=-\Gamma_2$$

$$2r\pi\, v=\text{const.}$$

$$v=\frac{\text{const.}}{r}$$

Thomson theorem

If the the integration path s is a fluid line of a perfect fluid, then $\frac{d\Gamma}{dt}=0$

Kutta-Joukowsky theorem: $F_{\text{lift}}=\rho\, v_{\infty}\Gamma$

Circulation is proportional wih the lift force.

How vorticity is produced?

Evolution of vorticity

Vorticity transport equation: $\nabla\times(\text{Navier}-\text{Stokes})$

Let's derive it in 2D! ω is a scalar in 2D: $\omega=\frac{\partial v}{\partial x}-\frac{\partial u}{\partial y}$

By taking the curl of N-S equation:

$$\frac{\partial}{\partial x}\left(\frac{\partial v}{\partial t}\right)+\frac{\partial}{\partial x}\left(u\frac{\partial v}{\partial x}\right)+\frac{\partial}{\partial x}\left(v\frac{\partial v}{\partial y}\right)=-\frac{1}{\rho}\frac{\partial}{\partial x}\left(\frac{\partial p}{\partial y}\right)+\frac{\partial g_y}{\partial x}+\nu\frac{\partial}{\partial x}\left(\frac{\partial^2 v}{\partial x^2}+\frac{\partial^2 v}{\partial y^2}\right)$$

$$\frac{\partial}{\partial y}\left(\frac{\partial u}{\partial t}\right)+\frac{\partial}{\partial y}\left(u\frac{\partial u}{\partial x}\right)+\frac{\partial}{\partial y}\left(v\frac{\partial u}{\partial y}\right)=-\frac{1}{\rho}\frac{\partial}{\partial y}\left(\frac{\partial p}{\partial x}\right)+\frac{\partial g_x}{\partial y}+\nu\frac{\partial}{\partial y}\left(\frac{\partial^2 u}{\partial x^2}+\frac{\partial^2 u}{\partial y^2}\right)$$

$$\frac{\partial\omega}{\partial t}+u\frac{\partial\omega}{\partial x}+v\frac{\partial\omega}{\partial y}+{*}{*}{*}=0+0+\nu\left(\frac{\partial^2\omega}{\partial x^2}+\frac{\partial^2\omega}{\partial y^2}\right)$$

$${*}{*}{*}=\frac{\partial u}{\partial x}\frac{\partial v}{\partial x}+\frac{\partial v}{\partial x}\frac{\partial v}{\partial y}-\frac{\partial u}{\partial y}\frac{\partial u}{\partial x}-\frac{\partial v}{\partial y}\frac{\partial u}{\partial y}=\frac{\partial v}{\partial x}\underbrace{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}_{0=}-\frac{\partial u}{\partial y}\underbrace{\left(\frac{\partial u}{\partial x}+\frac{\partial v}{\partial y}\right)}_{0=}$$

Because of the continuity:

Curl of the convective acceleration term in 3D flow

$$\vec{\omega}=\nabla\times\begin{pmatrix}u\\v\\w\end{pmatrix}=\begin{pmatrix}w'_y-v'_z\\u'_z-w'_x\\v'_x-u'_y\end{pmatrix}$$

$$\nabla\times\left(\vec{v}\cdot\nabla\vec{v}\right)=\nabla\times\left[\begin{pmatrix}u'_x & u'_y & u'_z\\ v'_x & v'_y & v'_z\\ w'_x & w'_y & w'_z\end{pmatrix}\begin{pmatrix}u\\v\\w\end{pmatrix}\right]=$$

$$=\begin{pmatrix}w''_{xy}-v''_{xz} & w''_{yy}-v''_{yz} & w''_{zy}-v''_{zz}\\ u''_{xz}-w''_{xx} & u''_{yz}-w''_{yx} & u''_{zz}-w''_{zx}\\ v''_{xx}-u''_{xy} & v''_{yx}-u''_{yy} & v''_{zx}-u''_{zy}\end{pmatrix}\begin{pmatrix}u\\v\\w\end{pmatrix}+$$

$$+\begin{pmatrix}w'_xu'_y-v'_xu'_z+w'_yv'_y-v'_yv'_z+w'_zw'_y-v'_zw'_z\\ u'_xu'_z-w'_xu'_x+u'_yv'_z-w'_yv'_x+u'_zw'_z-w'_zw'_x\\ v'_xu'_x-u'_xu'_y+v'_yv'_x-u'_yv'_y+v'_zw'_x-u'_zw'_y\end{pmatrix}$$

$$\vec{\omega}=\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}=\begin{pmatrix}w'_y-v'_z\\u'_z-w'_x\\v'_x-u'_y\end{pmatrix}$$

$$\nabla\times\left(\vec{v}\cdot\nabla\vec{v}\right)=$$

$$\begin{pmatrix}\alpha'_x & \alpha'_y & \alpha'_z\\ \beta'_x & \beta'_y & \beta'_z\\ \gamma'_x & \gamma'_y & \gamma'_z\end{pmatrix}\begin{pmatrix}u\\v\\w\end{pmatrix}+\begin{pmatrix}w'_xu'_y-v'_xu'_z-u'_xw'_y+u'_xv'_z\\ u'_yv'_z-w'_yv'_x-v'_yu'_z+v'_yw'_x\\ v'_zw'_x-u'_zw'_y-w'_zv'_x+w'_zu'_y\end{pmatrix}+\begin{pmatrix}\alpha\\ \beta\\ \gamma\end{pmatrix}\left(u'_x+v'_y+w'_z\right)$$

$$\vec{\omega}\cdot\nabla\vec{v}=\begin{pmatrix}u'_xw'_y-u'_xv'_z+\cancel{u'_y}u'_z-u'_yw'_x+u'_zv'_x-\cancel{w'_z}u'_y\\ v'_xw'_y-\cancel{v'_x}v'_z+v'_yu'_z-v'_yw'_x+\cancel{v'_z}v'_x-v'_zu'_y\\ \cancel{w'_x}w'_y-w'_xv'_z+w'_yu'_z-\cancel{w'_y}w'_x+w'_zv'_x-w'_zu'_y\end{pmatrix}$$

$$\nabla\times\left(\vec{v}\cdot\nabla\vec{v}\right)=\vec{v}\cdot\nabla\vec{\omega}-\vec{\omega}\cdot\nabla\vec{v}+\vec{\omega}\nabla\cdot\vec{v}$$

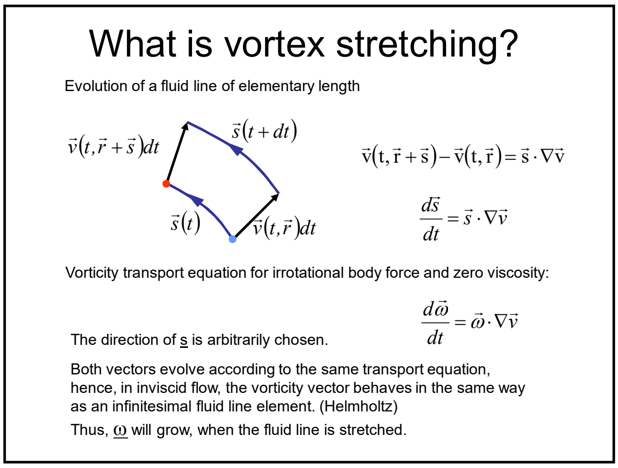
Vorticity transport

(ρ and ν are constants)

$$\frac{d\vec{v}}{dt}=\frac{\partial\vec{v}}{\partial t}+\vec{v}\cdot\nabla\vec{v}=-\frac{1}{\rho}\nabla p+\ \vec{g}\ +\ \nu\Delta\vec{v}\qquad /\qquad \nabla\times\ldots$$

$$\frac{d\vec{\omega}}{dt}=\frac{\partial\vec{\omega}}{\partial t}+\underbrace{\vec{v}\cdot\nabla\vec{\omega}}_{\text{vortex advection}}=\ 0\ +\underbrace{\nabla\times\vec{g}}_{\text{0,if g is irrotational}}+\underbrace{\nu\Delta\vec{\omega}}_{\text{vortex diffusion}}-\underbrace{\vec{\omega}\nabla\cdot\vec{v}}_{0}+\underbrace{\vec{\omega}\cdot\nabla\vec{v}}_{\text{vortex stretching}}$$

$$\frac{d\underline{\omega}}{dt}=\nu\Delta\underline{\omega}+\underline{\omega}\cdot\nabla\underline{v}$$



What is vortex stretching?

Evolution of a fluid line of elementary length

$$\vec{v}(t,\vec{r}+\vec{s})-\vec{v}(t,\vec{r})=\vec{s}\cdot\nabla\vec{v}$$

$$\frac{d\vec{s}}{dt}=\vec{s}\cdot\nabla\vec{v}$$

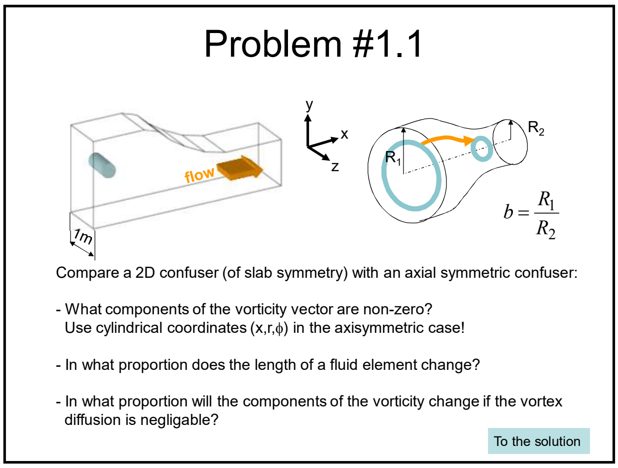
Vorticity transport equation for irrotational body force and zero viscosity:

$$\frac{d\vec{\omega}}{dt}=\vec{\omega}\cdot\nabla\vec{v}$$

The direction of $\underline{s}$ is arbitrarily chosen.

Both vectors evolve according to the same transport equation, hence, in inviscid flow, the vorticity vector behaves in the same way as an infinitesimal fluid line element. (Helmholtz)

Thus, $\underline{\omega}$ will grow, when the fluid line is stretched.



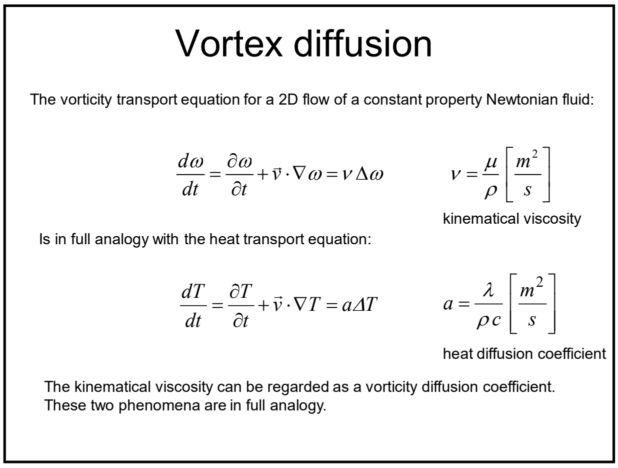
Problem #1.1

$$b=\frac{R_1}{R_2}$$

Compare a 2D confuser (of slab symmetry) with an axial symmetric confuser:

- What components of the vorticity vector are non-zero?
  Use cylindrical coordinates (x,r,ϕ) in the axisymmetric case!
- In what proportion does the length of a fluid element change?
- In what proportion will the components of the vorticity change if the vortex diffusion is negligable?

To the solution



Vortex diffusion

The vorticity transport equation for a 2D flow of a constant property Newtonian fluid:

$$\frac{d\omega}{dt}=\frac{\partial\omega}{\partial t}+\vec{v}\cdot\nabla\omega=\nu\Delta\omega \qquad \nu=\frac{\mu}{\rho}\left[\frac{m^2}{s}\right]$$

kinematical viscosity

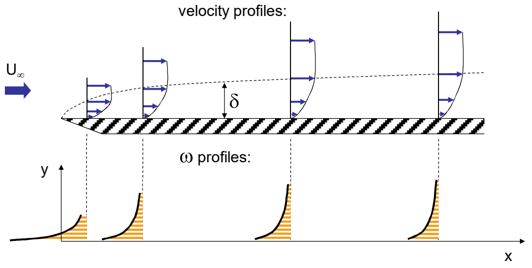
Is in full analogy with the heat transport equation:

$$\frac{dT}{dt}=\frac{\partial T}{\partial t}+\vec{v}\cdot\nabla T=a\Delta T \qquad a=\frac{\lambda}{\rho c}\left[\frac{m^2}{s}\right]$$

heat diffusion coefficient

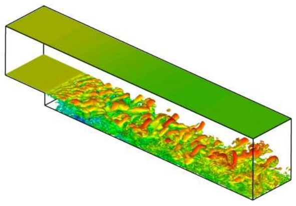
The kinematical viscosity can be regarded as a vorticity diffusion coefficient.
These two phenomena are in full analogy.

Boundary layer over a flat plate



Vorticity is continuously produced on the wall surface due to the no-slip condition, and it is conducted into the main stream by the viscosity.

The role of advection



http://www.computationalfluiddynamics.com.au/cfd-turbulence-part5-scale-resolving-simulations_srs/

Summary

The vorticity transport equation for incompressible fluids reads:

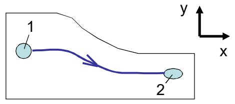
$$\frac{d\vec{\omega}}{dt} = \nabla \times \vec{g} + \nu \Delta \vec{\omega} + \vec{\omega} \cdot \nabla \vec{v}$$

Origin of vorticity:
- Boundary conditions (wall shear)
- Non conservative forces (eg. Coriolis force)

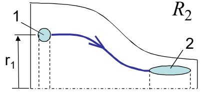
Redistribution of vorticity:
- Advection
- Vortex stretching
- Vortex diffusion

Solution

Solution of problem #1.1



$$b = \frac{R_1}{R_2} \cong \frac{r_1}{r_2}$$



$\omega_x = 0$ $\omega_y = 0$ $\omega_z \neq 0$

$s_{z,2} = s_{z,1} \longrightarrow \omega_{z,2} = \omega_{z,1}$

$\omega_x \neq 0$ $\omega_r = 0$ $\omega_\phi \neq 0$

$s_{\phi,2} = s_{\phi,1} b^{-1} \longrightarrow \omega_{\phi,2} = \omega_{\phi,1} b^{-1}$

from the velocity magnification:

$s_{x,2} = s_{x,1} b^2 \longrightarrow \omega_{x,2} = \omega_{x,1} b^2$