Turbulence and its modelling

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Outline

Definition and Properties of Turbulence

Properties
High Re number
Disordered, chaotic
3D phenomena
Unsteady
Continuum phenomena
Dissipative
Vortical
Diffusive
Continuous spatial spectrum
Has history

Notations
Summation convention
NS as example

Statistical description

Part I

First Lecture

Introduction

Why to deal with turbulence in a CFD course?

- Most of the equations considered in CFD are model equations
- Turbulence is a phenomena which is present in $\approx 95\%$ of CFD applications
- Turbulence can only be very rarely simulated and usually has to be modeled
- Basics of turbulence are required for the use of the models

Our limitations, simplifications

Following effects are not considered:

- density variation ($\rho = const.$)
    - Shock wave and turbulence interaction excluded
    - Buoyancy effects on turbulence not treated
- viscosity variation ($\nu = const.$)
- effect of body forces ($g_i = 0$)
    - Except free surface flows, gravity has no effect, can be merged in the pressure

Definition

Precise definition?

- No definition exists for turbulence till now
- Stability, chaos theory are the candidate disciplines to provide a definition
  - But the describing PDE's are much more complicated to treat than an ODE
- Last unsolved problem of classical physics ('Is it possible to make a theoretical model to describe the statistics of a turbulent flow?')
- Engineers still can deal with turbulence

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Instead of a definition

- Properties of turbulent flows can be summarized
- These characteristics can be used:
  - Distinguish between laminar (even unsteady) and turbulent flow
  - See the ways for the investigation of turbulence
  - See the engineering importance of turbulence

High Reynolds number

Reynolds number

- $Re = \frac{UL}{\nu} = \frac{F_{inertial}}{F_{viscous}}$
- high Re number $\longleftrightarrow$ viscous forces are small
- But inviscid flow is not turbulent

Role of Re

- Reynolds number is the bifurcation (stability) parameter of the flow
- The $Re_{cr} \approx 2300$ for pipe flows

Disordered, chaotic

- Terminology of dynamic systems
- Strong sensitivity on initial (IC) and boundary (BC) conditions
- Statement about the 'stability' of the flow
- PDE's (partial differential equations) have infinite times more degree of freedom (DoF) than ODE's (ordinary differential equations
  - Much more difficult to be treated
  - Can be the candidate to give a definition of turbulence
- The tool to explain difference between turbulence and 'simple' laminar unsteadiness

3D phenomena

- Vortex stretching (see e.g. Advanced Fluid Dynamics) is only present in 3D flows.
- In 2D there is no velocity component in the direction of the vorticity to stretch it.
- Responsible for scale reduction
- Responsible to vorticity enhancement

Averaged flow can be 2D

- Unsteady flowfield must be 3D
- The (Reynolds, time) averaged flowfield can be 2D
  - Spanwise fluctuations average to zero, but are required in the creation of streamwise, wall normal fluctuations

Unsteady

Turbulent flow is unsteady, but unsteadiness does not mean turbulence

Stability of the unsteady flow can be different

- In a unsteady laminar pipe flow (e.g. $500 < Re_b(t) < 1000$), the dependency on small perturbations is smooth and continuous
- In a unsteady turbulent pipe flow (e.g. $5000 < Re_b(t) < 5500$), the dependency on small perturbations is strong

Continuum phenomena

- Can be described by the continuum Navier-Stokes (NS) equations
- I.e. no molecular phenomena is involve as it it was

Conclusions

1. Can be simulated by solving the NS equations (Direct Numerical Simulation = DNS)
2. A smallest scale of turbulence exist, which is usually remarkable bigger than the molecular scales
3. The are cases, where molecular effects are important (re-entry capsule)
4. Turbulence is not fed from molecular resonations, but is a property (stability type) of the solution of the NS

Dissipative

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Dissipative

- Def: Conversion of mechanical (kinetic energy) to heat (raise the temperature)
- It is always present in turbulent flows
- It happens at small scales of turbulence, where viscous forces are important compared to inertia
- It is a remarkable difference to wave motion, where dissipation is not of primary importance

Vortical

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Turbulent flows are always vortical

- Vortex stretching is responsible for scale reduction
- Dissipation is active on the smallest scale

Diffusive

Diffusive property, the engineering consequence

- In the average turbulence usually increase transfers
  - E.g. friction factors are increased (e.g. λ)
  - Nusselt number is increased
- In the average turbulence usually increase transfer coefficients
  - Turbulent viscosity (momentum transfer) is increased
  - Turbulent heat conduction coefficient is increased
  - Turbulent diffusion coefficients are increased

Continuous spatial spectrum

Spatial spectrum

- Spatial spectrum is analogous to temporal one, defined by Fourier transformation
- Practically periodicity or infinite long domain is more difficult to find
- Visually: Flow features of every (between a bound) size are present

Counter-example

Acoustic waves have spike spectrum, with sub and super harmonics.

Has history, flow dependent, THE TURBULENCE does not exist

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Statistical description

As formulated in the *last unsolved problem of classical physics* no general rule of the turbulence could be developed till now.

No universality of turbulence has been discovered

- Turbulent flows can be of different type, e.g.:
  - It can be boundary condition dependent
  - It depends on upstream condition (spatial history)
  - It depends on temporal history

Notations

Directions

- x: Streamwise
- y: Wall normal, highest gradient
- z: Bi normal to x, y spanwise

Corresponding velocities

u, v, w

Index notation

$x = x_1,\ y = x_2,\ z = x_3$

$u = u_1,\ v = u_2,\ w = u_3$

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Notation (contd.)

Partial derivatives

$$\partial_j \stackrel{\text{def}}{=} \frac{\partial}{\partial x_j}$$

$$\partial_t \stackrel{\text{def}}{=} \frac{\partial}{\partial t}$$

Summation convention

Summation is carried out for double indices for the three spatial directions.

Very basic example

Scalar product:

$$a_i b_i \stackrel{\text{def}}{=} \sum_{i=1}^{3} a_i b_i \tag{1}$$

NS as example

Continuity eq.

$$\frac{\partial \rho}{\partial t} + div(\rho \mathrm{v}) = 0 \tag{2}$$

if $\rho = const.$, than

$$div \mathrm{v} = 0 \tag{3}$$

x component of the momentum eq.

$$\frac{\partial v_x}{\partial t} + v_x \frac{\partial v_x}{\partial x} + v_y \frac{\partial v_x}{\partial y} + v_z \frac{\partial v_x}{\partial z} = -\frac{1}{\rho}\frac{\partial p}{\partial x} + \nu \left(\frac{\partial^2 v_x}{\partial x^2} + \frac{\partial^2 v_x}{\partial y^2} + \frac{\partial^2 v_x}{\partial z^2} \right) \tag{4}$$

NS in short notation

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$\rho = const.$ continuity

$$\partial_i u_i = 0 \tag{5}$$

All the momentum equations

$$\partial_t u_i + u_j \partial_j u_i = -\frac{1}{\rho}\partial_i p + \nu \partial_j \partial_j u_i \tag{6}$$

Statistical description

The 'simple' approach

Turbulent flow can be characterised my its time average and the fluctuation compared to it

Problems of this approach

- How long should be the time average?
- How to distinguish between unsteadiness and turbulence?

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Flow examples

- Turbulent pipe flow having ($Re >> 2300$), driven by a piston pump (sinusoidal unsteadiness)
- Von Kármán vortex street around a cylinder of $Re = 10^5$, where the vortices are shedding with the frequency of $St = 0.2$

Difficult to distinguish between turbulence and unsteadiness

Ensemble average

Why to treat deterministic process by statistics?

- NS equations are deterministic (at least we believe, not proven generally)
- I.e. the solution is fully given by IC's and BC's
- Statistical description is useful because of the chaotic behaviour
  - The high sensitivity to the BC's and IC's
  - Possible to treat result of similar set of BC's and IC's statistically

Statistics

Solution as a statistical variable

$$\varphi = \varphi(x, y, z, t, i) \tag{7}$$

Index i corresponds to different but similar BC's and IC's

Density function

- Shows the 'probability' of a value of φ.

$$f(\varphi) \tag{8}$$

- It is normed:

$$\int_{-\infty}^{\infty} f(\varphi) \quad d\varphi = 1 \tag{9}$$

Mean value

Expected value

$$\overline{\varphi(x,y,z,t)} = \int_{-\infty}^{\infty} \varphi(x,y,z,t) \quad f\Big(\varphi(x,y,z,t)\Big) \quad d\varphi \qquad (10)$$

Average

$$\overline{\varphi(x,y,z,t)} = \lim_{N\to\infty} \frac{1}{N} \sum_{i=1}^{N} \varphi(x,y,z,t,i) \qquad (11)$$

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Reynolds averaging

Reynolds decomposition

Since the ensemble averaging is called Reynolds averaging, the decomposition is named also after Reynolds

$$\varphi = \overline{\varphi} + \varphi' \tag{12}$$

Fluctuation

$$\varphi' \stackrel{\text{def}}{=} \varphi - \overline{\varphi} \tag{13}$$

Properties of the averaging

Linearity

$$\overline{a\varphi + b\psi} = a\overline{\varphi} + b\overline{\psi} \tag{14}$$

Average of fluctuations is zero

$$\overline{\varphi'} = 0 \tag{15}$$

Properties of the averaging (contd.)

The Reynolds averaging acts only once

$$\overline{\overline{\varphi}} = \overline{\varphi} \tag{16}$$

Deviation

Deviation

- First characteristics of the fluctuations
- $$\sigma_\varphi = \sqrt{\overline{\varphi'^2}} \tag{17}$$
- Also called RMS: $\varphi_{rms} \stackrel{\text{def}}{=} \sigma_\varphi$

Connection between time and ensemble average

Ergodicity

Average is the same, deviation... ?

$$\hat{\varphi}^{(T)} = \frac{1}{T}\int_0^T \varphi \quad dt \tag{18}$$

$$\overline{\hat{\varphi}^{(T)}} = \frac{1}{T}\int_0^T \overline{\varphi} \quad dt = \overline{\varphi} \tag{19}$$

Correlations

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Covariance

$$\frac{R_{\varphi\psi}(x, y, z, t, \delta x, \delta y, \delta z, \tau) =}{\varphi'(x, y, z, t)\psi'(x + \delta x, y + \delta y, z + \delta z, t + \tau)}$$

Auto covariance

- If $\varphi = \psi$ covariance is called auto-covariance
- E.g. Time auto covariance:

$$R_{\varphi\varphi}(x, y, z, t, 0, 0, 0, \tau) \tag{20}$$

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Correlation

Non-dimensional covariance

$$\rho_{\varphi\psi}(x,y,z,t,\delta x,\delta y,\delta z,\tau) = \frac{R_{\varphi\psi}}{\sigma_{\varphi(x,y,z,t)}\sigma_{\psi(x+\delta x,y+\delta y,z+\delta z,t+\tau)}} \quad (21)$$

Integral time scale

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Integral time scale

$$T_{\varphi\psi}(x,y,z,t) = \int_{-\infty}^{+\infty} \rho_{\varphi\psi}(x,y,z,t,0,0,0,\tau) \quad d\tau \qquad (22)$$

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Taylor frozen vortex hypothesis

It is much more easy to measure the integral time scale (hot-wire) than the length scale (two hot-wire at variable distance)

Assumptions

- The flow field is completely frozen, characterised by the mean flow (U)
- The streamwise length scale can be approximated, by considering the temporal evolution of the frozen flowfield

Taylor approximated streamwise length scale

$$L^x = TU \tag{23}$$

Reynolds equations

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Statistical description

We will develop the Reynolds average of the NS equations, we will call the Reynolds equations

RA Continuity

The original equation

$$\partial_i u_i = 0$$

Development:

$$\begin{aligned}
\overline{\partial_i u_i} &= \\
&= \partial_i \overline{u_i} \\
&= \partial_i \overline{\overline{u_i} + u_i'} \\
&= \partial_i \overline{\overline{u_i}} \\
0 &= \partial_i \overline{u_i}
\end{aligned} \tag{24}$$

Same equation but for the average!

Momentum equations

Derivation

- Same rules applied to the linear term (no difference only)
- Non-linear term is different

Averaging of the non-linear term

$$
\begin{aligned}
\overline{u_j \partial_j u_i} &= \\
&= \overline{\partial_j (u_j u_i)} \\
&= \partial_j \overline{u_j u_i} \\
&= \partial_j \overline{(\overline{u_j} + u_j')(\overline{u_i} + u_i')} \\
&= \partial_j \left(\overline{\overline{u_j}\,\overline{u_i} + \overline{u_i}\, u_j' + \overline{u_j}\, u_i' + u_j' u_i'} \right) \\
&= \partial_j \left(\overline{\overline{u_j}\,\overline{u_i} + u_j' u_i'} \right) \\
&= \partial_j \left(\overline{u_j}\,\overline{u_i} \right) + \partial_j \overline{u_j' u_i'} \\
&= \overline{u_j}\, \partial_j \overline{u_i} + \partial_j \overline{u_i' u_j'}
\end{aligned}
\tag{25}
$$

Reynolds equations

Continuity equation

$$\partial_i \overline{u_i} = 0$$

Momentum equation

$$\partial_t \overline{u_i} + \overline{u_j}\,\partial_j \overline{u_i} = -\frac{1}{\rho}\partial_i \overline{p} + \nu \partial_j \partial_j \overline{u_i} - \partial_j \overline{u_i' u_j'} \qquad (26)$$

Reynold stress tensor

$$\overline{u_i' u_j'} \qquad (27)$$

Or multiplied by ρ, or -1 times

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Stresses

All stresses causing the acceleration

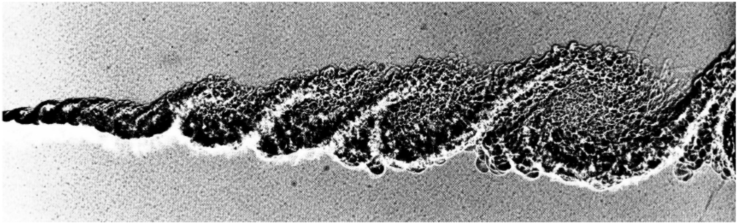
$$-\frac{1}{\rho}\overline{p}\,\delta_{ij} + \nu\partial_j\overline{u_i} - \overline{u_i'u_j'} \tag{28}$$

Part II

Second Lecture

Many scales of turbulence

Density variation visualise the different scales of turbulence in a mixing layer



Goal: Try to find some rules about the properties of turbulence at different scales

Kinetic energy

Kinetic energy:

$$E \stackrel{\text{def}}{=} \frac{1}{2} u_i u_i \tag{29}$$

Its Reynolds decomposition:

$$E = \frac{1}{2} u_i u_i = \frac{1}{2} (\overline{u_i}\,\overline{u_i} + 2u_i'\overline{u_i} + u_i'u_i') \tag{30}$$

Its Reynolds average

$$\overline{E} = \underbrace{\frac{1}{2} (\overline{u_i}\,\overline{u_i})}_{\hat{E}} + \underbrace{\frac{1}{2} (\overline{u_i'u_i'})}_{k} = \hat{E} + k \tag{31}$$

- The kinetic energy of the mean flow: $\hat{E}$
- The kinetic energy of the turbulence: k (Turbulent Kinetic Energy, TKE)

Richardson energy cascade

Vortex scales

High Re flow

- Typical velocity of the flow $\mathcal{U}$
- Typical length scale of the flow $\mathcal{L}$
- Corresponding Reynolds number ($\mathcal{R}e = \frac{\mathcal{U}\mathcal{L}}{\nu}$) is high

Turbulence is made of vortices of different sizes

Each class of vortex has:

- length scale: l
- velocity scale: $u(l)$
- time scale: $\tau(l) = l/u(l)$

Richardson energy cascade

The big scales

Biggest vortices

- size $l_0 \sim \mathcal{L}$
- velocity $u_0 = u_0(l_0) \sim u' = \sqrt{2/3k} \sim \mathcal{U}$

$\Rightarrow Re = \frac{u_0 l_0}{\nu}$ is also high

Fragmentation of the big vortices

- High *Re* corresponds to low viscous stabilisation
- Big vortices are unstable
- Big vortices break up into smaller ones

Richardson energy cascade

To the small scales

Inertial cascade

- As long as $Re(l)$ is high, inertial forces dominate, the break up continue
- At small scales $Re(l) \sim 1$ viscosity start to be important
  - The kinetic energy of the vortices dissipates into heat

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Scales of Turbulence

Transport equation of k

Modelling
Eddy Viscosity
Two equations models
Boundary Conditions
Inlet Boundary Conditions

The poem of Richardson

Big whorls have
little whorls that feed
on their velocity, and
little whorls have
smaller whorls and so
on to viscosity.

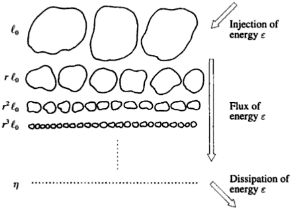
Lewis Fry Richardson F.R.S.

Richardson energy cascade

Connection between small and large scales

Dissipation equals production

- Dissipation is denoted by ε
- Because of the cascade can be characterised by large scale motion
- Dissipation: $\varepsilon \sim \frac{\text{kin. energy}}{\text{timescale}}$ @ the large scales
  - By formula: $\varepsilon = \frac{u_0^2}{l_0 / u_0} = \frac{u_0^3}{l_0}$



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NS symbol

For the description of development rules, it is useful to define the following NS symbol:

$$NS(u_i) \stackrel{\text{def}}{=} \partial_t u_i + u_j \partial_j u_i = \underbrace{-\frac{1}{\rho}\partial_i p + \nu \partial_j s_{ij}}_{\partial_j t_{ij}} \tag{32}$$

Let us repeat the development of the Reynolds equation!

$$\overline{NS(u_i)} \tag{33}$$

$$\partial_t \overline{u_i} + \overline{u_j}\,\partial_j \overline{u_i} = \partial_j \underbrace{\left[-\frac{1}{\rho}\overline{p}\,\delta_{ij} + \nu \overline{s}_{ij} - \overline{u_i' u_j'} \right]}_{\overline{T_{ij}}} \tag{34}$$

The TKE equation

Taking the trace of $\overline{(NS(u_i) - \overline{NS(u_i)})u_j'(NS(u_j) - \overline{NS(u_j)})u_i'}$

$$\partial_t k + \overline{u_j}\,\partial_j k = \underbrace{-a_{ij}\overline{s_{ij}}}_{\text{Production}} + \underbrace{\partial_j \left[\overline{u_j'\left(\frac{p'}{\rho} + k'\right)} - \nu \overline{u_i' s_{ij}'} \right]}_{\text{Transport}} - \underbrace{\varepsilon}_{\text{Dissipation}} \tag{35}$$

- Dissipation: $\varepsilon \overset{\text{def}}{=} 2\nu \overline{s_{ij}' s_{ij}'}$
- Anisotropy tensor: $a_{ij} \overset{\text{def}}{=} \overline{u_i' u_j'} - \frac{1}{3} \underbrace{\overline{u_l' u_l'}}_{2k} \delta_{ij}$

  Deviator part of the Reynolds stress tensor

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Production

- Expression: $\mathcal{P} \stackrel{\text{def}}{=} -a_{ij}\overline{s_{ij}}$
- Transfer of kinetic energy from mean flow to turbulence
  - The same term with opposite sign in the equation for kin. energy of mean flow
- The mechanism to put energy in the "Richardson" cascade
- Happens at the large scales

The TKE equation

Meaning of the terms (contd.)

Dissipation

- Expression: $\varepsilon \stackrel{\text{def}}{=} 2\nu \overline{s'_{ij} s'_{ij}}$
- Conversion of kinetic energy of turbulence to heat
  - Work of the viscous stresses at small scale (s'_{ij})
- The mechanism to draw energy from the "Richardson" cascade
- Happens at the small scales

$\mathcal{P} = \varepsilon$ if the turbulence is homogeneous (isotropic), as in the "Richardson" cascade

The TKE equation

Meaning of the terms (contd.)

Transport

- Expression: $\partial_j \left[\overline{u'_j \left(\frac{p'}{\rho} + k' \right)} - \nu \overline{u'_i s'_{ij}} \right]$
- Transport of turbulent kinetic energy in space
  - The expression is in the form of a divergence $(\partial_j \square_j)$
  - Divergence can be reformulated to surface fluxes (G-O theorem)

Idea of RANS modelling

- Solving the Reynolds averaged NS for the averaged variables $(\overline{u}, \overline{v}, \overline{w}, \overline{p})$
- The Reynolds stress tensor $\overline{u_i' u_j'}$ is unknown and has to be modelled
- Modelling should use the available quantities $(\overline{u}, \overline{v}, \overline{w}, \overline{p})$

Usefulness

- If the averaged results are useful for the engineers
- i.e. the fluctuation are not interesting "only" their effect on the mean flow
- If modelling is accurate enough

Eddy Viscosity modell

Idea

- Effect of turbulence is similar to effect of moving molecules in kinetic gas theory
- The exchange of momentum between layers of different momentum is by the perpendicularly moving molecules
- Viscous stress is computed by: $\Phi_{ij} = 2\nu S_{ij}$

Eddy Viscosity modell (contd.)

In equations...

- Only the deviatoric part is modelled
- The trace (k) can be merged to the pressure (modified pressure), and does not need to be modelled
- Modified pressure is used in the pressure correction methods to satisfy continuity (see Poisson eq. for pressure)

$$\overline{u_i' u_j'} - \frac{2}{3} k \delta_{ij} = -2 \nu_t \overline{S_{ij}} \quad (36)$$

Eddy Viscosity

Viscosity is a product of a length scale (l') and a velocity fluctuation scale (u')

- The length scale has to be proportional to the distance, what the fluid part moves by keeping its momentum
- The velocity fluctuation scale should be related to the velocity fluctuation caused by the motion of the fluid part

$$\nu_t \sim l' u' \tag{37}$$

Newer results supporting the concept

Coherent structure view of turbulence, proves that there are fluid parts (vortices) which keep their properties for a while, when moving

Two equations models

- Length (l') and velocity fluctuation scales (u') are properties of the flow and not the fluid, they are changing spatially and temporally
- PDE's for describing evolutions are needed

Requirements for the scales

- Has to be well defined
- Equation for its evolution has to be developed
- Has to be numerically "nice"
- Should be measurable easily to make experimental validation possible

k-e modell

Velocity fluctuation scale

- TKE is characteristic for velocity fluctuation
- It is isotropic (has no preferred direction)

$$u' \sim \sqrt{k} \tag{38}$$

Length scale

- Integral length scale is well defined (see correlations)
- No direct equation is easy to develop
- Length scale is computed through the dissipation

Recall: $\varepsilon = \frac{u_0^3}{l_0} \Rightarrow l' \sim \frac{k^{3/2}}{\varepsilon}$

Equation for the eddy viscosity

$$\nu_t = C_\nu \frac{k^2}{\varepsilon} \tag{39}$$

C_ν is a constant to be determined by theory or experiments...

Our status...?

- We have two unknown (k, ε) instead of one (ν_t)

k model equation

Equation for k was developed, but there are unknown terms:

$$\partial_t k + \overline{u_j}\,\partial_j k = \underbrace{-a_{ij}\overline{s_{ij}}}_{\text{Production}} + \underbrace{\partial_j \left[\overline{u_j'\left(\frac{p'}{\rho} + k'\right)} - \nu \overline{u_i' s_{ij}'} \right]}_{\text{Transport}} - \underbrace{\varepsilon}_{\text{Dissipation}} \tag{40}$$

Production

Production is directly computable, by using the eddy viscosity hypothesis

$$\mathcal{P} = -a_{ij}\overline{S_{ij}} = 2\nu_t \overline{S_{ij}}\,\overline{S_{ij}} \tag{41}$$

k model equation

Dissipation

Separate equation will be derived

Transport $\partial_j T_j$

- Can be approximated by gradient diffusion hypothesis

$$T_j = \frac{\nu_t}{\sigma_k} \partial_j k \tag{42}$$

- σ_k is of Schmidt number type to rescale ν_t to the required diffusion coeff.
  - To be determined experimentally

Summarised k model equation

$$\partial_t k + \overline{u_j}\,\partial_j k = 2\nu_t \overline{S_{ij}}\,\overline{S_{ij}} - \varepsilon - \partial_j\left(\frac{\nu_t}{\sigma_k}\partial_j k\right) \tag{43}$$

- Everything is directly computable (except ε)
- The LHS is the local and convective changes of k
  - Convection is an important property of turbulence (it is appropriately treated by these means)

Model equation for ε

- It is assumed that it is described by a transport equation
- Instead of derivation, based on the k equation

$$\partial_t \varepsilon + \overline{u_j}\, \partial_j \varepsilon = C_{1\varepsilon} \mathcal{P} \frac{\varepsilon}{k} - C_{2\varepsilon} \varepsilon \frac{\varepsilon}{k} - \partial_j \left(\frac{\nu_t}{\sigma_\varepsilon} \partial_j \varepsilon \right) \tag{44}$$

- Production and dissipation are rescaled ($\frac{\varepsilon}{k}$) and "improved" by constant coefficients ($C_{1\varepsilon}$, $C_{2\varepsilon}$)
- Gradient diffusion for the transport using Schmidt number of σ_ε
- The ε equation is not very accurate! :)

Constants of the standard k-e model

$$C_\nu = 0,09 \quad (45)$$

$$C_{1\varepsilon} = 1,44 \quad (46)$$

$$C_{2\varepsilon} = 1,92 \quad (47)$$

$$\sigma_k = 1 \quad (48)$$

$$\sigma_\varepsilon = 1,3 \quad (49)$$

Example for the constants

Homogeneous turbulence

$$\mathrm{d}_t k = \mathcal{P} - \varepsilon \tag{50}$$

$$\mathrm{d}_t \varepsilon = C_{1\varepsilon} \mathcal{P} \frac{\varepsilon}{k} - C_{2\varepsilon} \varepsilon \frac{\varepsilon}{k} \tag{51}$$

Example for the constants

Decaying turbulence

Since $\mathcal{P} = 0$, the system of equations can be solved easily:

- $k(t) = k_0 \left(\frac{t}{t_0} \right)^{-n}$
- $\varepsilon(t) = \varepsilon_0 \left(\frac{t}{t_0} \right)^{-n-1}$
- $n = \frac{1}{C_{2\varepsilon} - 1}$
- n is measurable "easily"

k-ω modell

- k equation is the same
- $\omega \stackrel{\text{def}}{=} \frac{1}{C_\nu} \frac{\varepsilon}{k}$ Specific dissipation, turbulence frequency (ω)
- equation for ω similarly to ε equation
  - transport equation, with production, dissipation and transport on the RHS
- ω equation is better close to walls
- ε equation is better at far-field

$\Rightarrow$ SST model blends the two type of length scale equation, depending on the wall distance

Required Boundary Conditions

The turbulence model PDE's are transport equations, similar to the energy equation

- Local change
- Convection
- Source terms
- Transport terms

Inlet Boundary Conditions

- Neumann or Dirichlet or mixed type of BC can be used generally
- Inlet is usually Dirichlet (specified value)

Final goal

- How to prescribe k and ε or ω at inlet boundaries?

Approximation of inlet BC's

To use easy quantities, which can be guessed

Develop equations to compute k and ε or ω from quantities, which can be guessed by engineers

Turbulence intensity

$$Tu \stackrel{\text{def}}{=} \frac{u'}{\overline{u}} = \frac{\sqrt{2/3k}}{\overline{u}}$$

Approximation of inlet BC's

Length scale

$l' \sim \frac{k^{3/2}}{\varepsilon} \Rightarrow \varepsilon$

- From measurement (using Taylor hypothesis)
- Law of the wall (later)
- Guess from hydraulic diameter $l \approx 0.07 d_H$

Importance of inlet BC's

If turbulence is governing a flow

- Example: Atmospheric flows, where geometry is very simple (flat land, hill) turbulence is complex
  - by spatial history of the flow
  - over rough surface
  - including buoyancy effects
- Sensitivity to turbulence at the inlet has to be checked
  - the uncertainty of the simulation can be recognised
  - measurement should be included
  - the simulation domain should be extended upstream

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Part III

Third Lecture

Wall boundary conditions

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- Both k and ε or ω require boundary conditions at the walls
- Before introducing the boundary conditions and the approximate boundary treatment techniques, some theory about wall boundary layers is required

Channel flow

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Characteristics

- Flow between two infinite plates $\Rightarrow$ fully developed
- Channel half width: δ
- Bulk velocity: $U_b \stackrel{\text{def}}{=} \frac{1}{\delta}\int_0^\delta \overline{u}\,\mathrm{d}y$
- Bulk Reynolds number: $Re_b \stackrel{\text{def}}{=} \frac{U_b 2\delta}{\nu}$
- $Re_b > 1800$ means turbulence

Channel flow (contd.)

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Streamwise averaged momentum equation:

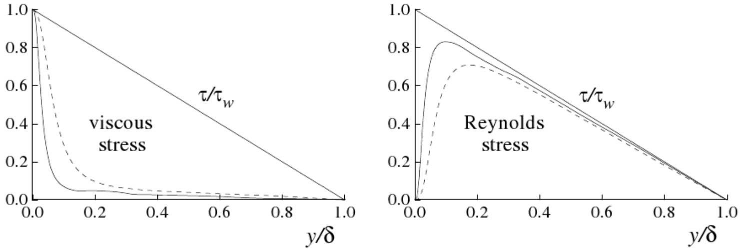
$$0 = \underbrace{\nu \mathrm{d}^2_{y^2} \overline{u}}_{\mathrm{d}_y \tau_l} - \underbrace{\mathrm{d}_y \overline{u'v'}}_{\mathrm{d}_y \tau_t} - \frac{1}{\rho} \partial_x \overline{p} \tag{52}$$

The pressure gradient ($\mathrm{d}_x \overline{p_w}$) is balanced by the two shear stresses: $\tau = \tau_l + \tau_t$
Its distribution is linear:

$$\tau(y) = \tau_w \left(1 - \frac{y}{\delta}\right) \tag{53}$$

Channel flow (contd.)

Two type of shear stresses



The two shear stresses

- The viscous stress is dominant at the wall
- Turbulent stress is dominant far from the wall
- Both stresses are important in an intermediate region

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Two scales of the flow at the wall

Definitions

- Friction velocity: $u_\tau \stackrel{\text{def}}{=} \sqrt{\frac{\tau_w}{\rho}} = \sqrt{-\frac{\delta}{\rho} \mathrm{d}_x \overline{p_w}}$
- Friction Reynolds number: $Re_\tau \stackrel{\text{def}}{=} \frac{u_\tau \delta}{\nu} = \frac{\delta}{\delta_\nu}$
- Viscous length scale: $\delta_\nu \stackrel{\text{def}}{=} \frac{u_\tau}{\nu}$

General law of the wall can be characterised:

$$\mathrm{d}_y \overline{u} = \frac{u_\tau}{y} \Phi\left(\frac{y}{\delta_\nu}, \frac{y}{\delta}\right) \tag{54}$$

Φ is a function to be determined!

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It can be assumed that only the wall scale is playing in the wall proximity:

$$\mathrm{d}_y \overline{u} = \frac{u_\tau}{y} \Phi_I \left(\frac{y}{\delta_\nu} \right) \qquad \text{for } y \ll \delta \tag{55}$$

Wall non-dimensionalisation $\Box^+$

$$u^+ \overset{\text{def}}{=} \frac{\overline{u}}{u_\tau} \tag{56}$$

$$y^+ \overset{\text{def}}{=} \frac{y}{\delta_\nu} \tag{57}$$

Law of the wall

Velocity

Viscous sub-layer

- Only τ_l is counting
- $u^+ = y^+$
- for $y^+ < 5$

Logarithmic layer

- Viscosity is not in the scaling
- $\Phi_l = \frac{1}{\kappa}$ for $y \ll \delta$ and $y^+ \gg 1$
- Log-law: $u^+ = \frac{1}{\kappa} \ln(y^+) + B$
  - From measurements: $\kappa \approx 0.41$ and $B \approx 5.2$

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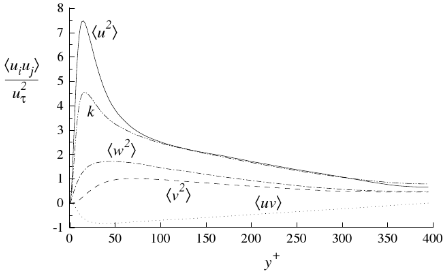
Law of the wall
Velocity

Outer layer

- Φ depends only on y/δ
- In CFD we want to compute it for the specific cases! $\Rightarrow$ We do not deal with it.

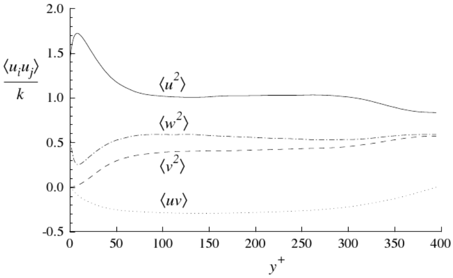
Reynolds stress tensor at the wall

u_τ scaling



Sharp peaks around $y^+ = 20$

Reynolds stress tensor at the wall
k scaling



A platau is visible in the log law region.

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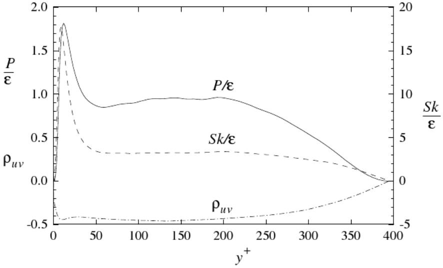
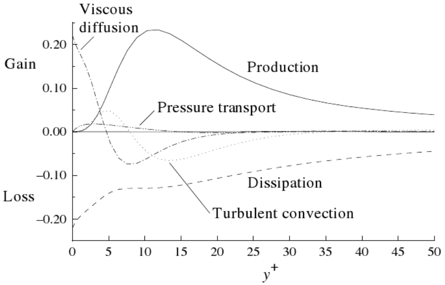
TKE budget at the wall



- $\mathcal{P}/\varepsilon \approx 1$ in the log-law region
- $\mathcal{P}/\varepsilon \approx 1.8$ close to the wall

TKE budget at the wall



- Turbulence is mainly produced in the buffer region ($5 < y^+ < 30$)
- Turbulence is viscous diffused to the wall
- Turbulence is strongly dissipated at the wall
- Conclusion: $\varepsilon = \nu \mathrm{d}^2_{y^2} k \quad @ \; y = 0$

Numerical treatment of the wall layer, actual BC's

Low Re treatment

In this treatment the complete boundary layer is resolved numerically

When to do?

- Low Reynolds number flow, where resolution is feasible
- If boundary layer is not simple, can not be described by law of the wall

How to do?

- Use a turbulence model incorporating near wall viscous effects
- Use appropriate wall resolution ($y^+ < 1$)

Numerical treatment of the wall layer, actual BC's

High Re treatment

In this treatment the first cell incorporates the law of the wall

When to do?

- High Reynolds number flow, where it is impossible to resolve the near wall region
- If boundary layer is simple, can be well described by law of the wall

How to do?

- Use a turbulence models containing law of the wall BC
- Use appropriate wall resolution ($30 < y^+ < 300$)

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The mixture of the two methods is developed:

- to enable the engineer not to deal with the wall resolution
- usually the mixture of the two method is needed, depending on actual position in the domain

Resolution requirements

At any kind of treatment the boundary layer thickness (δ) has to resolved by ≈ 20 cells to ensure accuracy.

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Simulation

In the simulation the turbulence phenomena is actually resolved by a numerical technique, by solving the describing equations

Modelling

In the modelling of turbulence the effects of turbulence are modelled relying on theoretical and experimental knowledge. In the computation a reduced description of turbulence is carried out

Direct Numerical Simulation = DNS

The NS equations (describing completely the turbulence phenomena) are solved numerically

Difficulties

- The scales where the dissipation is effective are very small
  - The size of the smallest scales are Reynolds number dependent
- Simulation is only possible for academic situations (e.g.: HIT on 64 $\cdot 10^9$ cells)

Concept of LES

Compromise between RANS and DNS

- RANS: feasible but inaccurate
- DNS: accurate but infeasible

The large scales are import to simulate

- The large scales of the turbulent flow are boundary condition dependent, they needs to be simulated
- The small scales of turbulence are more or less universal and can be modelled ‘easily’
- The removal of the small scales form the simulation reduce the computational cost remarkably

Filtering

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How to develop the equations?
How to separate between large and small scales?

Spatial filtering, smoothing using a kernel function

$$\langle\varphi\rangle(x_j,t) \stackrel{\text{def}}{=} \int_V G_\Delta(r_i;x_j) \quad \varphi(x_j - r_i, t)\mathrm{d}r_i \tag{58}$$

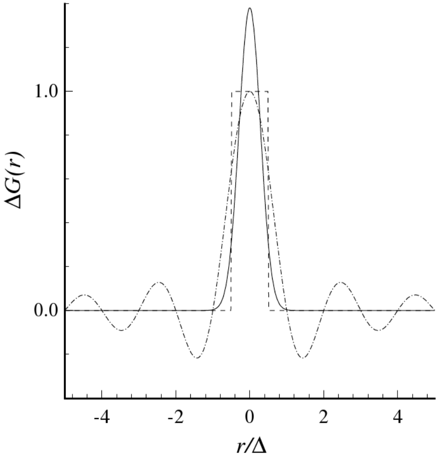
Filtering kernel

- G_Δ is the filtering kernel with a typical size of Δ.
- G_Δ has a compact support (its definition set where the value is non-zero is closed) in its first variable
- To be the filtered value of a constant itself it has to be true:

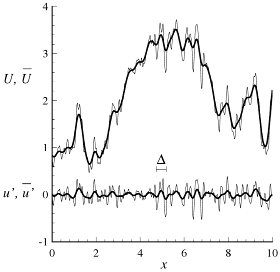
$$\int_V G_\Delta(r_i; x_j)\mathrm{d}r_i = 1 \tag{59}$$

- If $G_\Delta(r_i; x_j)$ is homogeneous in its second variable and isotropic in its first variable than $G_\Delta(|r_i|)$ is a function of only one variable

Filtering kernel
Examples



Filtering
Physical space

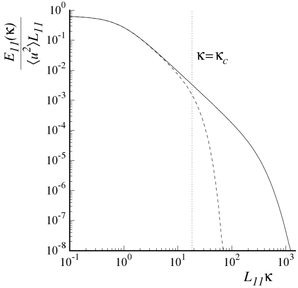


Fluctuation:

$$\widetilde{\varphi} \stackrel{\text{def}}{=} \varphi - \langle \varphi \rangle \tag{60}$$

$\langle \widetilde{\varphi} \rangle \neq 0$, a difference compared to Reynolds averaging

Filtering
Spectral space



Recall: the cutting wavenumber (κ_c), below which modelling is needed

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Filtered equations

- If using the previously defined (homogeneous, isotropic) filter
- Averaging and the derivatives commute (exchangeable)

$$\partial_i \langle u_i \rangle = 0 \quad (61)$$

$$\partial_t \langle u_i \rangle + \langle u_j \rangle \partial_j \langle u_i \rangle = -\frac{1}{\rho} \langle p \rangle + \nu \partial_j \partial_j \langle u_i \rangle - \partial_j \tau_{ij} \quad (62)$$

- 3D (because turbulence is 3D)
- unsteady (because the large eddies are unsteady)

Sub Grid Scale stress

τ_{ij} is called Sub-Grid Scale stress SGS from the times when filtering was directly associated to the grid

$$\tau_{ij} \stackrel{\text{def}}{=} \langle u_i u_j \rangle - \langle u_i \rangle \langle u_j \rangle \tag{63}$$

- It represents the effect of the filtered scales
- It is in a form a stress tensor
- Should be dissipative to represent the dissipation on the filtered small scale

Eddy viscosity model

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- Same as in RANS
- $$\tau_{ij} - \frac{1}{3}\tau_{kk}\delta_{ij} = -2\nu_t \langle s_{ij} \rangle \quad (64)$$
- Relatively a better approach since the small scales are more universal
- Dissipative if $\nu_t > 0$.

Smagorinsky model

- $$\nu_t = (C_s\Delta)^2 |\langle S\rangle| \tag{65}$$
- $$|\langle S\rangle| \stackrel{\text{def}}{=} \sqrt{2s_{ij}s_{ij}} \tag{66}$$
- C_s Smagorinsky constant to b determined
  - using spectral theory of turbulence
  - using validations on real flow computations
- Δ to be prescribed
  - Determine the computational cost (if too small)
  - Determine the accuracy (if too big)
  - 80% of the energy is resolved is a compromise

Scale Similarity model

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Let us assume that the cuted small scales are similar to the kept large scales!
A logical model:

$$\tau_{ij} \stackrel{\text{def}}{=} \langle\langle u_i \rangle \langle u_j \rangle\rangle - \langle\langle u_i \rangle\rangle \langle\langle u_j \rangle\rangle \tag{67}$$

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Properties

- It is not dissipative enough
- It gives feasible shear stresses (from experience)
- Logical to combine with Smag. model!

Dynamic approach

- The idea is the same as in the scale similarity model
- The theory is more complicated
- Any model can be made dynamic
- Dynamic Smagorinsky is widely used (combining the two advantages)

Boundary Conditions
Periodicity

- Periodicity is used to model infinite long domain
- The length of periodicity is given by the length scales of turbulence

Boundary Conditions
Inlet

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- Much more difficult than in RANS
- Turbulent structures should be represented
  - Vortices should be synthesized
  - Separate precursor simulation to provide “real” turbulence

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$$y^+ \approx 1 \tag{68}$$

$$x^+ \approx 50 \tag{69}$$

$$z^+ \approx 10 - 20 \tag{70}$$