Large-Eddy Simulation in Mechanical Engineering

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Outline

1 Geurts Book
This lecture is strongly based on the book:

Bernard J. Geurts:
Elements of Direct and Large-Eddy Simulation
http://www.rtedwards.com/books/072/index.html
Part I

Second Lecture
The idea

Reduced flow description
Low-pass filtering (in wave number space), with characteristic length $\Delta$
Filtering of non-linear terms $\iff$ Closure problems
Important to consider the ratio $\Delta/\eta$.
As $\Delta/\eta \rightarrow O(1)$, LES converge to DNS.
$\Delta/\eta$ is the main parameter of the LES. It characterise the resolution/quality of various ”results” of the simulation.
The governing equations

Incompressible, isochoric fluid $\rho = \text{const}$. Constant viscosity $\nu = \text{const}$.

Continuity equation:

$$\partial_i u_i = 0 \quad (1)$$

$$\partial_t u_i + u_j \partial_j u_i = -\partial_i \frac{\rho}{\rho} + \nu \partial_j \partial_j u_i \quad (2)$$

Let us assume that $\rho = 1$ so we can skip it from the equations.
Transformation properties of the equations

Galilean invariance etc.
Notation of the filtering

Let us define an operator $L$ which reduces the spatial scales of the flow-field:

$$L : \varphi \rightarrow \langle \varphi \rangle$$

1D example ($\varphi(x, t)$ or e.g. $u(x, t)$):

$$
\langle \varphi \rangle = L(\varphi)(x, t) = \int_{x - \Delta/2}^{x + \Delta/2} \varphi(\xi, t) \, d\xi
$$  \hspace{1cm} (3)

Two filtering is shown: 1) $\Delta = l/4$, 2) $\Delta = l/16$
Preliminary comment on filter width

We are interested in a quantitative variable $q$ of a specific flow (for example flow around cylinder, channel flow). The filtered flow-field can be used also for the evaluation. Depending on the interesting quantity (drag coefficient, turbulence level), the accuracy of the estimation $q_{err}$ will clearly depend on the applied filter length $\Delta$ i.e. how many details of the flow we keep.

For some quantity the very smoothed signal ($\Delta$ is big) gives already a good prediction ($q_{err}$ is small), for other quantities fines scales are also responsible and only filtering at smaller scales can be applied.
Notation of the filtering

In 3D a generic homogeneous filter can be defined:

\[ \langle \varphi \rangle (\mathbf{x}, t) = L(\varphi)(\mathbf{x}, t) = \int_{-\infty}^{+\infty} G_\Delta(\mathbf{x}' - \mathbf{x}) \varphi(\mathbf{x}', t) \, d\mathbf{x}' = G_\Delta * \varphi \quad (4) \]

This a convolution filter (multiplied by something shifted over)!
No temporal filtering is carried out!

- Causality: future does not influence the past
- Temporal averaging would lead us to RANS

Normalization of the kernel:

\[ \int_{-\infty}^{+\infty} G_\Delta(\mathbf{x}') \, d\mathbf{x}' = 1 \quad (5) \]
This type of filtering is a linear integral operator. The above defined homogeneous convolution filter commutes both with temporal and spatial derivation:

\[
\langle \partial_t f \rangle = \partial_t \langle f \rangle \quad ; \quad \langle \partial_i f \rangle = \partial_i \langle f \rangle
\] (6)
Commuting Filtering of the mom. eq.

Because of the commutation property: \((\rho = 1 \text{ and everything is rearranged to the LHS)}\)

\[
\partial_t \langle u_i \rangle + \partial_j (\langle u_i u_j \rangle) + \partial_i \langle p \rangle - \nu \partial_j \partial_j \langle u_i \rangle = 0 \quad (7)
\]

If we want to see the same terms on LHS as in the orig. NS. we have to add the required term \(\partial_j (\langle u_i \rangle \langle u_j \rangle)\) and subtract the one available \(\partial_j (\langle u_i u_j \rangle)\) on both side of the equation:

\[
\partial_t \langle u_i \rangle + \partial_j (\langle u_i \rangle \langle u_j \rangle) + \partial_i \langle p \rangle - \nu \partial_j \partial_j \langle u_i \rangle = -\partial_j \tau_{ij} \quad (8)
\]

where:

\[
\tau_{ij} \overset{\text{def}}{=} \langle u_i u_j \rangle - \langle u_i \rangle \langle u_j \rangle \quad (9)
\]
\( \tau_{ij} \) is called the SubGrid Scale stress tensor. It is responsible for the effect of the unresolved (removed by the filter) scales. Its actual effect is \( \partial_j \tau_{ij} \) which needs to be modelled.

It is called historically sub grid scale, since the filtering was originally (but not very logically) related to the numerical grid size \( (h) \).
SGS tensor as a commutator

Let us define the product operator:

\[ \Pi(u_i, u_j) \overset{\text{def}}{=} u_i u_j \]  

(10)

The SGS stress can be written using this as a commutator:

\[ \tau_{ij} = \langle u_i u_j \rangle - \langle u_i \rangle \langle u_j \rangle = \]

\[ = L(\Pi(u_i, u_j)) - \Pi(L(u_i), L(u_i)) = \]

\[ = [L, \Pi](u_i, u_j) \]  

(11)

where \([a, b] \overset{\text{def}}{=} ab - ba\) is the commutator symbol for spaces where addition and multiplication are defined. It measures the strength of not commuting.
Identities of the commutator operator

Wikipedia section

1. 
   \[ [A, A] = 0 \]

2. 
   \[ [A, B] = -[B, A], \text{ anti-commutativity} \]

3. 
   \[ [A, [B, C]] + [B, [C, A]] + [C, [A, B]] = 0, \text{ Jacobi identity} \]

4. 
   \[ [A, BC] = [A, B]C + B[A, C], \text{ Leibnitz rule} \]

Example for Leibnitz rule:

\[ A \overset{\text{def}}{=} d_x(*) = (*)' \]
\[ B \overset{\text{def}}{=} f(*) \]
\[ C \overset{\text{def}}{=} g(*) \]

(12)

I.e.: \((fg)' = f'g + fg'\)
Non-uniform filtering

In convolution filtering the filter width ($\Delta$) is constant. Practically the mesh size is changing (e.g.: see channel in wall normal direction)!

1D example ($\varphi(x, t)$ or e.g. $u(x, t)$):

$$\langle \varphi \rangle = L(\varphi)(x, t) = \int_{x-\Delta_-(x)}^{x+\Delta_+(x)} \frac{g(x, \xi)}{\Delta(x)} \varphi(\xi, t) \, d\xi$$

(13)

where $\Delta(x) = \Delta_- + \Delta_+$ is the spatially varying filter width.
Commutation error at non-uniform filtering

\[ \langle \partial_i \varphi \rangle \neq \partial_i \langle \varphi \rangle \] because of the varying filter width. This can be written more formally:

\[ L(\partial_i \varphi) - \partial_i L(\varphi) = [L, \partial_i](\varphi) \neq 0 \]  \hspace{1cm} (14)
Non-uniformly filtered set of equations

Continuity

The continuity equation:

\[ \partial_j \langle u_j \rangle = -[L, \partial_j](u_j) \]  \hspace{1cm} (15)

i.e. the filtered flow-field is not any more divergence free (solenoidal).

Variation of the filter width is related to local mass production or destruction.
Non-uniformly filtered set of equations
Momentum

The momentum equation:

\[
\frac{\partial}{\partial t} \langle u_i \rangle + \partial_j (\langle u_i \rangle \langle u_j \rangle) + \partial_i \langle p \rangle - \nu \partial_j \partial_j \langle u_i \rangle = -\partial_j ([L, \Pi](u_i, u_j)) \\
([L, \partial_j](p) - \nu[L, \partial_{jj}](u_i) + [L, \partial_j](\Pi(u_i, u_j))
\]

(16)

The RHS is much more complicated now.
The new terms even destroy the conservation property of the original equations.
Flow structures travelling in the direction of filter width change experience a scale reduction or occurrence simply by the change in the filter.
Non-uniformly filtered set of equations
Momentum

E.g. a vortex arriving to a region of higher filter width will be smoothed corresponding to local kinetic energy reduction, which is against conservation. The effect can be quantified by material derivative of the filter width: $\langle u_j \rangle \partial_j \Delta$
Summary

We arrive at the 'LES-template':

\[
\text{NS}(\langle U \rangle) = R(U, \langle U \rangle)
\]

(17)

where \( R \) is the residual because of the filtering. \( R(U, U) = 0 \), since in this case \( L = Id \) identity operator, and DNS should be recovered. In the modelling 'exercise' \( R(U, \langle U \rangle) \hookrightarrow M(\langle U \rangle) \), i.e. the residual has to be approximated using the filtered flow-field.
How to develop the equations?
How to separate between large and small scales?

Spatial filtering, smoothing using a kernel function

\[
\langle \varphi \rangle (x_j, t) \overset{\text{def}}{=} \int_V G_\Delta(r_i; x_j) \varphi(x_j - r_i, t) dr_i
\]  

(18)
Filtering kernel

- $G_\Delta$ is the filtering kernel with a typical size of $\Delta$.
- $G_\Delta$ has a compact support (its definition set where the value is non-zero is closed) in its first variable.
- To be the filtered value of a constant itself it has to be true:
  \[ \int_V G_\Delta(r_i; x_j)dr_i = 1 \] (19)
- If $G_\Delta(r_i; x_j)$ is homogeneous in its second variable and isotropic in its first variable than $G_\Delta(|r_i|)$ is a function of only one variable.
Filtering kernel
Examples
Fluctuation:

\[ \tilde{\varphi} \overset{\text{def}}{=} \varphi - \langle \varphi \rangle \]  

\[ \langle \varphi \rangle \neq 0, \text{ a difference compared to Reynolds averaging} \]
Recall: the cutting wavenumber ($\kappa_c$), below which modelling is needed
Filtered equations

- If using the previously defined (homogeneous, isotropic) filter
- Averaging and the derivatives commute (exchangeable)

\[ \partial_i \langle u_i \rangle = 0 \]  \hspace{1cm} (21)

\[ \partial_t \langle u_i \rangle + \langle u_j \rangle \partial_j \langle u_i \rangle = - \frac{1}{\rho} \langle p \rangle + \nu \partial_j \partial_j \langle u_i \rangle - \partial_j \tau_{ij} \]  \hspace{1cm} (22)

- 3D (because turbulence is 3D)
- unsteady (because the large eddies are unsteady)
Sub Grid Scale stress

\( \tau_{ij} \) is called Sub-Grid Scale stress SGS from the times when filtering was directly associated to the grid

\[
\tau_{ij} \overset{\text{def}}{=} \langle u_i u_j \rangle - \langle u_i \rangle \langle u_j \rangle \tag{23}
\]

- It represents the effect of the filtered scales
- It is in a form of a stress tensor
- Should be dissipative to represent the dissipation on the filtered small scale
Eddy viscosity model

- Same as in RANS

\[ \tau_{ij} - \frac{1}{3} \tau_{kk} \delta_{ij} = -2 \nu_t \langle s_{ij} \rangle \] (24)

- Relatively a better approach (compared to the application for RANS) since the small scales are more universal

- Dissipative if \( \nu_t > 0 \).
Smagorinsky model

\[ \nu_t = (C_s \Delta)^2 |\langle S \rangle| \quad (25) \]

\[ |\langle S \rangle| \overset{\text{def}}{=} \sqrt{2s_{ij}s_{ij}} \quad (26) \]

- \( C_s \) Smagorinsky constant to be determined
  - using spectral theory of turbulence
  - using validations on real flow computations
- \( \Delta \) to be prescribed
  - determine the computational cost (if too small)
  - determine the accuracy (if too big)
  - resolving 80% of the energy is a good compromise
Let us assume that the cuted small scales are similar to the kept large scales!
A logical model:

\[ \tau_{ij} \stackrel{\text{def}}{=} \langle \langle u_i \rangle \langle u_j \rangle \rangle - \langle \langle u_i \rangle \rangle \langle \langle u_j \rangle \rangle \]  

(27)

The unfiltered quantities are modelled by the filtered ones.

Properties

- It is not dissipative enough
- It gives feasible shear stresses (from experience)
- Logical to combine with Smag. model!
Dynamic approach

- The idea is the same as in the scale similarity model
- The theory is more complicated
- Any model can be made dynamic
- Dynamic Smagorinsky is widely used (combining the two advantages)
Boundary Conditions

Periodicity

- Periodicity is used to model infinite long domain
- The length of periodicity is given by the length scales of turbulence
Boundary Conditions
Inlet

- Much more difficult than in RANS
- Turbulent structures should be represented
  - Vortices should be synthesized
  - Separate precursor simulation to provide “real” turbulence
Boundary Conditions

Wall

\begin{alignat}{3}
y^+ & \approx 1 \quad & (28) \\
x^+ & \approx 50 \quad & (29) \\
z^+ & \approx 10 - 20 \quad & (30)
\end{alignat}